REGULAR NUMBER OF COMPLEMENTARY PRISM OF FERRERS GRAPH

[1]R. Chenthil ThangaBama and [2]S. Sujitha

[1]Register Number 18113132092001, Research Scholar,
Department of Mathematics,
Manonmaniam Sundaranar University,Tirunelveli,India.
email: chenthilthangabama@gmail.com
[2]Assistant Professor, Department of Mathematics,
Holy Cross College (Autonomous), Nagercoil – 629 004, India.
email: sujitha.s@holycrossngl.edu.in
Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli – 627 012, Tamil Nadu, India

Abstract

The regular number r(G) of a graph G is the minimum number of subsets into which the edge set of G is partitioned so that the subgraph induced by each subset is regular. In this paper, we examine the regular number r(G) of complementary prism of a Ferrers graph.

AMS Subject Classification: 05C50, 05C25.

Keywords: Ferrers graph, complementary prism, regular number, diameter, radius.

DOI: 10.48047/ecb/2022.11.12.123

1. Introduction

Graph theory notation and terminology are not given here we refer it from [1]. The complement of a graph G is a graph $\overline{G}$ on the same set of vertices as of G such that there will be an edge between two vertices in $\overline{G}$, if and only if there is no edge in between in G. The complementary prism of a graph G, denoted by $G\overline{G}$, as the graph formed from the disjoint union of G and its complement $\overline{G}$ by adding the edges of the perfect matching between the corresponding vertices of G and $\overline{G}$, Where $V\left(G\overline{G}\right)=V(G)\cup V(\overline{G})$. The regular number of G is defined to be the minimum number of subsets into which the edge set of G can be partitioned so that the subgraph induced by each subset is regular.

In this paper we find the regular number of complementary prism of a Ferrers graph. Also we find the diameter and radius of the complementary prism of a Ferrers graph.

Theorem 1.1 [2] If G = (V, E) is a Ferrers graph iff for all distrinct x, y, z, w $\in$ E then d(x, w) + d(y, z) $\leq$ 4.

Theorem 1.2 [5] For any path P_n, $r(P_n) = 2$.

Theorem 1.3 [5] For any graph G, r(G) = 1 if and only if G is regular.

Theorem 1.4 [1] For any complete bipartite graph $K_{3,n}$, where $n \geq 1$, $r(K_{3,n}) = \frac{n}{3}$ if $n \equiv 0$ (mod 3), and $r(K_{3,n}) = \left\lfloor \frac{n}{3} \right\rfloor + 3$ if $n \equiv 1, 2$ (mod 3)

Theorem 1.5 [3] If G is a Ferrers graph, then $d(u,v) \leq 3$ for all $u, v \in V(G)$.

Theorem 1.6 [4] For a graph G, G is a Ferrers tree if and only if G has two internal vertices.

2. Main results

Theorem 2.1. For any path $G = P_n$, $G\bar{G}$ is a Ferrers graph for n = 2 and non-Ferrers graph otherwise.

Proof. Case (i) When n = 2

In this case $G\bar{G} = P_4$, and by Theorem 1.6, P_4 is a Ferrers graph.

Case (ii) When n ≥ 3

Consider a path P_n with n (≥ 3) vertices. Let $x_1, x_2, ...,x_n$ be the vertices of P_n and $\overline{x_1}, \overline{x_2}, ..., \overline{x_n}$ be the vertices of $\overline{P_n}$. Consider the four vertices $x_1, \overline{x_1}, x_{n-1}, x_n$ in $G\overline{G}$. By the definition of complementary prism, $\overline{x_1}$ is adjacent to $x_1, \bar{x}_2, \bar{x}_3, , \dots\dots, \bar{x}_n$ Hence $d(x_1, x_n) = 3$ and $d(\overline{x_1}, x_{n-1}) = 2$ (or) $d(x_1, x_{n-1}) = 2$ and $d(\overline{x_1}, x_n) = 3.$ In both the cases, $d(x_1, x_n) + d(\overline{x_1}, x_{n-1}) > 4$ and $d(x_1, x_{n-1}) + d(\overline{x_1}, x_n) > 4.$ Therefore, by Theorem 1.1, $G\overline{G}$ is a non-Ferrers graph.

Theorem 2.2. For any path P_2, the regular number $r(P_2\overline{P_2}) = 2$

Proof. Consider P_2. Clearly $P_2\overline{P}_2 = P_4$, which is a path. Therefore by Theorem 1.2, $r(P_2\overline{P_2}) = r(P_4) = 2.$

Theorem 2.3. For a cycle $G = C_n$, $G\overline{G}$ is a Ferrers graph for n = 3 and non-Ferrers for n ≥ 4.

Proof. When n = 3

Clearly C_3 is an infringe Ferrers graph and $\overline{C_3}$ is a null graph. Let x, y, z be the vertices of C_3 and $\bar{x}, \bar{y}, \bar{z}$ be the vertices of $\overline{C_3}$. Consider any two disjoint edges $x\bar{x}$ and $y\bar{y}$ in $C_3\overline{C_3}$. Then we find $d(x, \bar{y}) + d(\bar{x}, y) = 4.$ Hence by Theorem 1.1, $C_3\overline{C_3}$ is Ferrers graph.

When n ≥ 4

Consider a cycle C_n with n (≥ 4) vertices. Let $x_1, x_2, ..., x_n$ be the vertices of C_n and $\overline{x_1}, \overline{x_2}, ..., \overline{x_n}$ be the vertices of $\overline{C_n}$. Consider the four vertices $\overline{x_1}, \overline{x_3}, x_4, \overline{x_4}$ in $G\overline{G}$. Clearly $d(\overline{x_1}, \overline{x_4}) = 3$ and $d(\overline{x_3}, x_4) = 2$ (or) $d(\overline{x_1}, x_4) = 2$ and $d(\overline{x_3}, \overline{x_4}) = 3$. In both the cases $d(\overline{x_1}, \overline{x_4}) + d(\overline{x_3}, x_4) > 4$ and $d(\overline{x_1}, x_4) + d(\overline{x_3}, \overline{x_4}) > 4$. By theorem 1.1, $G\overline{G}$ is non-Ferrers.

Theorem 2.4. For any complementary prism of C_3, the regular number $r(C_3\overline{C_3}) = 2$

Proof. Consider the complementary prism of C_3. To prove $r(C_3\overline{C_3}) = 2$. Suppose $r(C_3\overline{C_3}) \neq 2$. Let x, y, z, $\overline{x}, \overline{y}, \overline{z}$ be the vertices in $C_3\overline{C_3}$ and xy, yz, xz, $x\overline{x}, y\overline{y}, z\overline{z}$ be the edges in $C_3\overline{C_3}$ and is shown in Figure 1.

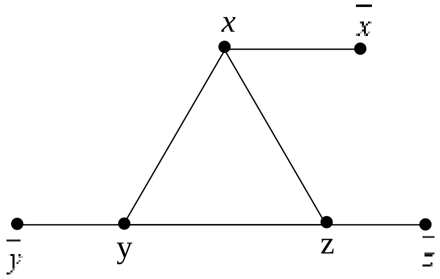

Figure 1

Clearly $C\overline{C_3}$ is not a regular graph. Hence r($C_3\overline{C_3}$) ≠ 1.Also $C_3\overline{C_3}$ contains one C_3 cycle and the remaining non adjacent edges are in one set.Hence $r(C_3\overline{C_3}) = 2$

Theorem 2.5, For a wheel graph $G = W_n$, $G\overline{G}$ is ferrers for n = 4 and non-Ferrers for n ≥ 5.

Proof**.** Let $G = W_n$ be a wheel graph on n vertices.

When n = 4

Clearly W_4 is a regular graph with degree 3, and $\overline{W_4}$ is a null graph. Let x, y, z, w be the vertices in W_4 and $\overline{x}, \overline{y}, \overline{z}, \overline{w}$ be the vertices in $\overline{W_4}$. By the definition of complementary prism $xy, yz, xz, xw, yw, zw, x\overline{x}, y\overline{y}, z\overline{z}, w\overline{w}$ are the edges in $W_4\overline{W_4}$. Consider any two non adjacent edges $x\overline{x}, z\overline{z} \in W_4\overline{W_4}$. Then $d(x, \overline{z}) + d(\overline{x}, z) = 4$. for every $x, \overline{x}, z$ and $\overline{z} \in V(W_4\overline{W_4})$. Therefore by Theorem 1.1, $W_4\overline{W_4}$ is a Ferrers graph.

When n ≥ 5

Consider a wheel graph W_n with n (≥ 5) vertices. Let $x_1, x_2, \ldots, x_n$ be the vertices of W_n and $\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}$ be the vertices in $\overline{W_n}$. Let z be the center vertex of W_n. Clearly deg (z) = n-1 and the remaining vertices have degree 3.Also $\bar{z}$ is an isolated vertex in $\overline{W_n}$ and the remaining vertices have degree n – 4 in $\overline{W_n}$, $\overline{W_n}$ is a disconnected graph. But by the definition of complementary prism $W_n\overline{W_n}$ is a connected graph. Consider the four vertices x_1, x_2, x_4, $\overline{x_4}$ in $G\overline{G}$. Clearly $d(x_1, \overline{x_4}) = 3$ and $d(x_2, x_4) = 2$. Then we find $d(x_1, \overline{x_4}) + d(x_2, x_4) > 4$. By theorem 1.1, $G\overline{G}$ is non-Ferrers.

Theorem 2.6 For any complementary prism of a wheel graph W_4, the regular number $r(W_4\overline{W_4}) = 2$

Proof Consider the wheel G = W_4. By Theorem 2.5, $G\overline{G}$ is Ferrers. To prove that $r(W_4\overline{W_4}) = 2$. Let v_1, v_2, v_3, v_4, $\overline{v_1}, \overline{v_2}, \overline{v_3}, \overline{v_4}$ be the vertices in $W_4\overline{W_4}$ and v_1v_2, v_2v_3, v_1v_3, v_1v_4, v_2v_4, v_3v_4, $v_1\overline{v_1}, v_2\overline{v_2}, v_3\overline{v_3}$ be the edges in $W_4\overline{W_4}$ and is shown in Figure 2.

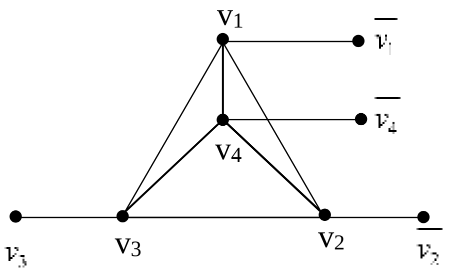

Figure 2

Case(i) Suppose $r(W_4\overline{W_4}) = 1$. By Theorem 1.3, $W_4\overline{W_4}$ is a regular graph.Which is a contradiction to $W_4\overline{W_4}$ is not regular. Hence $r(W_4\overline{W_4}) \neq 1$.

Case(ii) Suppose $r(W_4\overline{W_4}) > 2$. By Theorem 1.4, $W_4\overline{W_4}$ is a complete bipartite graph $K_{3,n}$ where n ≥ 1. From figure 2, $W_4\overline{W_4}$ is not a complete bipartite graph. Which is a contradiction. Hence $r(W_4\overline{W_4}) \ngtr 2$. Hence in both the cases $r(W_4\overline{W_4}) \neq 1$ and $r(W_4\overline{W_4}) \ngtr 2$. Thus $r(W_4\overline{W_4}) = 2$.

Theorem 2.7 The complementary prism of a complete graph is a Ferrers graph.

Proof Consider a complete graph $G = K_n$ with n vertices. And $\overline{K_n}$ is a complement of K_n which is a null graph. But by the definition of complementary prism $G\overline{G}$ is a connected graph. Let $x_1, x_2, x_3 \ldots, x_n$ be the vertices of K_n and $\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}$ be the vertices of $\overline{K_n}$. Consider any four vertices $x_1, \overline{x_1}, x_n, \overline{x_n}$ in $G\overline{G}$. Clearly $d(x_1, \overline{x_n}) = 2$ and $d(\overline{x_1}, x_n) = 2$ (or) $d(x_1, x_n) = 3$ and $d(\overline{x_1}, x_n) = 1$. In both the cases, $d(x_1, \overline{x_n}) + d(\overline{x_1}, x_n) \leq 4$. Therefore,by Theorem 1.1, $G\overline{G}$ is Ferrers.

Theorem 2.8. For any Complementary prism of K4, the regular number $r(K_n\overline{K_n}) = 2$

Proof. Consider a Ferrers graph $K_n\overline{K_n}$ To prove that $r(K_n\overline{K_n}) = 2$. Let v_1, v_2, v_3, v_4, $\overline{v_1}, \overline{v_2}, \overline{v_3}, \overline{v_4}$ be the vertices in $K_n\overline{K_n}$ and v_1v_2, v_2v_3, v_1v_3, v_1v_4, v_2v_4, v_3v_4, $v_1\overline{v_1}, v_2\overline{v_2}, v_3\overline{v_3}$ be the edges in $K_n\overline{K_n}$ and is shown in Figure 3.

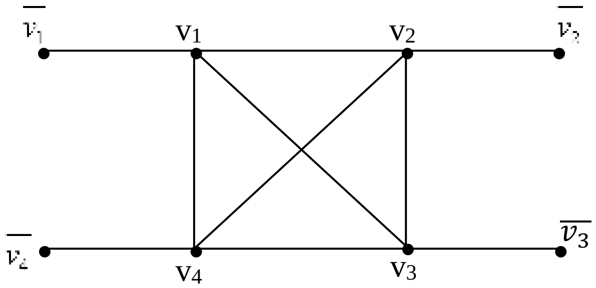

Figure 3

Case (i) Suppose $r(K_n\overline{K_n}) = 1$. By theorem 1.3, $K_n\overline{K_n}$ is a regular graph. But $K_n\overline{K_n}$ is not a regular graph. Which is a contradiction. Hence $r(K_n\overline{K_n}) \neq 1$.

Case (ii) Suppose $r(K_n\overline{K_n}) > 2$. By theorem 1.4, $K_n\overline{K_n}$ is a complete bipartite graph $K_{3,n}$ where $n \geq 1$. From figure 3, $K_n\overline{K_n}$ is not a complete bipartite graph. Which is a contradiction. Hence $r(K_n\overline{K_n}) \ngtr 2$. Hence in both the cases $r(K_n\overline{K_n}) \neq 1$ and $r(K_n\overline{K_n}) \ngtr 2$.Thus $r(K_n\overline{K_n}) = 2$.

Theorem 2.9. Let $G=K_{m,n}$ be a complete bipartite graph on m+n vertices. Then $G\overline{G}$ is non-Ferrrers for every $m, n \geq 2$

Proof. Consider a complete bi-partite graph $K_{m,n}$ with m, n (≥2) vertices. Let V_1 & V_2 be the two partitions of G = $K_{m,n}$. Let $x_1, x_2, x_3 ...., x_m$ be the vertices of V_1 and $y_1, y_2, y_3 ...., y_n$ be the vertices of V_2. Also $\overline{x_1}, \overline{x_2}, \overline{x_3} ...., \overline{x_m}$ and $\overline{y_1}, \overline{y_2}, \overline{y_3} ...., \overline{y_n}$ be the vertices of $\overline{K}_{m,n}$. Consider the four vertices $\overline{x_1}, \overline{x_m}, \overline{y_1}, \overline{y_n}$ in $K_{m,n}\overline{K_{m,n}}$. Clearly $d(\overline{x_1}, \overline{y_n}) = 4$ and $d(\overline{x_m}, \overline{y_1}) = 3$ (or) $d(\overline{x_1}, \overline{y_1}) = 3$ and $d(\overline{x_m}, \overline{y_n}) = 3$. In both the cases, $d(\overline{x_1}, \overline{y_n}) + d(\overline{x_m}, \overline{y_1}) > 4$.Therefore by Theoren 1.1, $G\overline{G}$ is non -Ferrers.

Theorem 2.10 For a complete bipartite graph $K_{m,n}$, $r(K_{m,n}\overline{K_{m,n}}) = 3$

Proof. To prove, $r\left(K_{m,n}\,\overline{K_{m,n}}\right)=3$. Suppose $r\left(K_{m,n}\,\overline{K_{m,n}}\right)\neq 3$.

Case (i). Suppose $r\left(K_{m,n}\,\overline{K_{m,n}}\right)=1$, then $K_{m,n}\,\overline{K_{m,n}}$ is a regular graph (or) P_2 (or) K_n. But $K_{m,n}\,\overline{K_{m,n}}$ is not a regular graph. Hence $r\left(K_{m,n}\,\overline{K_{m,n}}\right)\neq 1$.

Case (ii). Suppose $r\left(K_{m,n}\,\overline{K_{m,n}}\right)=2$, then $K_{m,n}\,\overline{K_{m,n}}$ is either a path containing atleast 3 vertices (or) $p_2\overline{p_2}$ or $C_3\overline{C_3}$ (or) $W_4\overline{W_4}$ (or) $K_n\overline{K_n}$. In all the cases the graph is Ferrers. Which is a contradiction. Hence $r\left(K_{m,n}\,\overline{K_{m,n}}\right)\neq 2$

Case (iii). Suppose that $K_{m,n}\,\overline{K_{m,n}} > 3$, then $K_{m,n}\,\overline{K_{m,n}}$ is either a wheel graph with atleast 6 vertices (or) $K_{3,3}$. In all the cases the graph is a Ferrers graph. Which is a contradiction. Hence $r\left(K_{m,n}\,\overline{K_{m,n}}\right)=3$

Theorem 2.11. For a tree G, $G\overline{G}$ is non Ferrers except path P_2 (n = 2).

Proof.

We Prove the result by the following two cases.

Case (i) Suppose G is a Ferrers tree.

To prove that $G\overline{G}$ is a non Ferrers graph. Since G is a Ferrers tree. By Theorem 1.6. G has 2 internal vertices. Let us assume that x_1, x_2 be the internal vertices and $x_3, x_5, x_7, ...,x_{n-1}$ be the vertices adjacent with x_1 and $x_2, x_4, x_6, ...,x_n$ be the vertices adjacent with x_2. Also, $\overline{x_1}, \overline{x_2}, \overline{x_3}, ..., \overline{x_{n-1}}, \overline{x_n}$ are the vertices in $\overline{G}$. Consider the four vertices $x_{n-1}, \overline{x_{n-1}}, x_2, x_n$ in $G\overline{G}$. Clearly $d(x_n, x_{n-1})=3$ and $d(x_2, \overline{x_{n-1}})=2$. Then we find $d(x_n, x_{n-1})+ d(x_2, \overline{x_{n-1}})>4$. By Theorem 1.1, $G\overline{G}$ is non Ferrers.

Case (ii) Suppose G is a non-Ferrers tree.

To prove that $G\overline{G}$ is a non-Ferrers graph. Since G is a non-Ferrers tree. Let $x_1, x_2, x_3 ....,x_n$ be the vertices of G, and $\overline{x_1}, \overline{x_2}, \overline{x_3} ....,\overline{x_n}$ be the vertices of $\overline{G}$. Consider the four vertices $x_{n-1}, x_n, x_1, \overline{x_1}$ in $G\overline{G}$. Clearly $d(x_{n-1}, x_1)=3$ and $d(x_n, \overline{x_1})=3$. In both the cases $d(x_{n-1}, x_1)+d(x_n, \overline{x_1})>4$. By Theorem 1.1, $G\overline{G}$ is non Ferrers.

Theorem 2.12. Let G be any graph, $G\overline{G}$ is Ferrers, then diam ($G\overline{G}$) = 3 and rad ($G\overline{G}$) = 2.

Proof. Let G be any graph and $G\overline{G}$ is a Ferrers graph. To prove that, diam ($G\overline{G}$) = 3 and rad $(G\overline{G}) = 2$ since $G\overline{G}$ is a Ferrers g raph, by Theorem 1.5, d (u,v) ≤ 3 for all u, v ϵ V($G\overline{G}$). Then $G\overline{G}$ graph attains the upper bound value. Hence diam ($G\overline{G}$) = 3. Now to prove that rad $(G\overline{G}) = 2$. It is enough to prove that rad $(G\overline{G}) \neq 3$ and rad $(G\overline{G}) \neq 1$.

Case (i) Suppose $rad(G\overline{G}) = 3$, then diam $(G\overline{G}) \geq 3$. If diam $(G\overline{G}) > 3$, then by Theorem 2.12, $G\overline{G}$ is not a Ferrers graph. Which is a contradiction. Hence $r(G\overline{G}) \neq 3$

Case (ii) Suppose $rad(G\overline{G}) = 1$, then $G\overline{G}$ is a regular graph. By Theorem 2.12, $G\overline{G}$ is not a regular graph. Hence $rad(G\overline{G}) \neq 1$. Hence in both the cases $rad(G\overline{G}) \neq 3$ and $rad(G\overline{G}) \neq 1$. Thus $rad(G\overline{G}) = 2$.

3. Conclusion.

In this paper we proved that the regular number of the complementary prism of a Ferrers graph is 2. Also we have seen that the diameter and radius of the complementary prism of a Ferrers graph.

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