Commutative Monoids on Bounded Sum and Bounded Product over Bipolar Fuzzy Matrices

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Abstract: In our real life, the bipolar fuzzy theory is a core feature to be considered positive information represents what is possible or preferred, while negative information represents what is forbidden or surely false. In this paper, we introduce the bounded sum and bounded product operations on bipolar fuzzy matrices and study the properties. We intend to construct an algebraic structure by proving various algebraic properties on bounded sum and bounded product over bipolar fuzzy matrices. Finally, we construct a commutative monoid on a bounded sum and a bounded product over bipolar fuzzy matrices.

Keywords and Phrases: Bipolar Fuzzy Matrix (BFM), Bipolar Fuzzy Set (BFS), Bounded sum, Bounded product, Commutative monoid.

INTRODUCTION

In 1965, Zadeh [16] introduced the concept of fuzzy set theory; after that, many authors generalized it. One ideal generalization is the intuitionistic fuzzy theory by Atanassov [1,2], in which he introduced some operators and found a promising direction of research. Masaharu Mizummoto and Kokichi Tanaka [8] investigated the algebraic properties of fuzzy sets as well as the properties of fuzzy sets. Fuzzy matrices were defined for the first time by Thomson in 1977 who discussed the convergence of the power of a fuzzy matrix. Many authors, including Shyamal and Pal [14], Khan and Pal [4] and Xin [15] studied controllable fuzzy matrices, fuzzy algebra, and its matrix theory with max-min operation.

A bipolar fuzzy set theory is invertible for dealing with both polarity and fuzziness. In 1994 Zhang [17] introduced an extension of the fuzzy set named the Bipolar Fuzzy Set (BFS). Samantha and Pal introduced the bipolar fuzzy hypergraphs and bipolar fuzzy intersection graphs. Madhumangal Pal and Sanjib Mondal [10] introduced the bipolar fuzzy matrices and introduced some basic properties of bipolar fuzzy elements by using max-min composition and the properties of transitive closure and power-convergent are investigated with examples, Lee J.G. and Hur K. [6] studied bipolar fuzzy relations.

In addition to these operations, new operations called "bounded sum" and "bounded difference" are introduced by Zadeh and Masaharu Mizummoto to investigate fuzzy reasoning problems that are too complex for a precise solution. Silambarasan and Sriram [13] studied the bounded sum and bounded product of fuzzy matrices. Muthuraji and Sriram [9] investigated commutative monoids and monoid homomorphism on Lukasiwicz conjunction and disjunction operators on Intuitionistic Fuzzy Matrices. A bipolar fuzzy matrix is essential to solve or model a problem relating to a bipolar uncertain system. In this paper, we explore some more algebraic results of the said operators and discuss the relationships between other operators.

PRELIMINARIES

We recollect some relevant basic definitions and results will be used later.

Definition 2.1[6]: A fuzzy set A in a universe of discourse U is characterized by a membership function μ_A which takes the values in the unit interval [0, 1] ie, $\mu_A : U \to [0,1]$. The value of μ_A at u($\in U$), $\mu_A(u)$, represents the grade of membership (grade for short) of u in A and is a point in [0, 1].

Definition 2.2: Let A be an $n \times m$ matrix defined by $A = \begin{bmatrix} a_{11}a_{12} & . & . & . & a_{1m} \\ a_{21}a_{22} & . & . & . & a_{2m} \\ . & . & . & & . \\ . & . & . & & . \\ . & . & . & & . \\ a_{n1}a_{n2} & . & . & . & a_{nm} \end{bmatrix}$

The matrix A is a fuzzy matrix if and only if $a_{ij} \in [0,1]$ for $1 \le i \le n$ and $1 \le j \le m$. In otherwords, any $n \times m$ matrix A is a fuzzy matrix if the elements of A are in the interval $[0,1]$.

Definition 2.3[10]: A $BFS\ B_F$ in X (universe of discourse) is an object having the form $B_F = \left\{\left(x, \mu_{\underline{n}}(x), \mu_{\overline{p}}(x)\right)\right\}$ where $\mu_{\underline{n}}: X \to [-1,0]$ and $\mu_{\overline{p}}: X \to [0,1]$ are two mappings. The positive membership degree $\mu_{\overline{p}}(x)$ denotes the satisfaction degree of an element x to the property corresponding to a $BFS\ B_F$ and the negative membership degree $\mu_{\underline{n}}(x)$ denotes the satisfaction degree of x to some implicit counter property of B_F .

Definition 2.4[10]: Let X,Y $\epsilon\ B_F$ where $X = (-x_n, x_p)$ and $Y = (-y_n, y_p)$ and $x_n, x_p, y_n, y_p\ \epsilon[0,1]$. Then

i. X+Y= $(-[x_n \vee y_n], [x_p \vee y_p])$
ii. X .Y= $(-[x_n \wedge y_n], [x_p \wedge y_p])$
iii. X $\le$ Y if and only if $x_n \le y_n$ and $x_p \le y_p$ That is, X $\le$ Y if and only if X+Y = Y.
iv. The complement of X is denoted by X^c and is defined by
$$X^c = (-x_n, x_p)^c = (-1 + x_n, 1 - x_p)$$
v. The zero element of a BFS is denoted by O_b and is defined by $O_b = (0,0)$.
vi. The unit element of a BFS is denoted by i_b and is defined by $i_b = (-1,1)$.
vii. The identity element of a BFS is denoted by e_b and is defined by $e_b = (0,1)$.

Definition 2.5[10]: Let $A = [a_{ij}]_{l \times m} \in M_{lm}$, then $a_{ij} = (-a_{ijn}, a_{ijp}) \in$ B$_F$, where $a_{ijn}, a_{ijp} \in [0,1]$ are the negative and positive membership values of the element a_{ij} respectively.The set of all rectangular BFMs of order $l \times m$ is denoted by M_{lm} and that of square BFMs of order $m \times m$ is denoted by M_m.

Definition 2.6[10]: Let A = $[a_{ij}]$, B = $[b_{ij}] \in M_{lm}$ be two $BFMs$. Therefore, $a_{ij}, b_{ij} \in$ B$_F$, then

i. $A + B = [a_{ij} + b_{ij}]_{l \times m} = (-\max\{a_{ijn}, b_{ijn}\}, \max\{a_{ijp}, b_{ijp}\})_{l \times m}$
ii. $A \cdot B = [a_{ij} . b_{ij}]_{l \times m} = (-\min\{a_{ijn}, b_{ijn}\}, \min\{a_{ijp}, b_{ijp}\})_{l \times m}$
iii. $A \le B$ if and only if $a_{ijn} \le b_{ijn}$ and $a_{ijp} \le b_{ijp}$ That, A $\le$ B if and only if $A + B = B$.

iv. The complement of A is denoted by A^c and is defined by

$$A^c = (-a_{ijn}, a_{ijp})^c == (-1 + a_{ijn}, 1 - a_{ijp})$$

v. The zero matrix O_m of order $m \times m$ is the matrix where all the elements are $O_b = (0, 0)$

vi. The identity matrix I_m of order $m \times m$ is the matrix where all the diagonal entries are $i_b = (-1, 1)$ and all other entries are $O_b = (0, 0)$.

vii. A is symmetric if only if $(a_{ij}, a_{ij'}) = (a_{ji}, a_{ji'})$ for all i, j.

Definition 2.7[9]: Let M be a fixed set. Let $e_* \in M$ be a unitary element of M and Let $*$ be an operation. Then $(M, *, e_*)$ is a commutative monoid if

i. For all $x, y \in M$ implies $x * y \in M$

ii. For all $x, y, z \in M$ implies $(x * y) * z = x * (y * z)$

iii. For all $x, \in M$, $x * e_* = x = e_* * x$

iv. For all $x, y \in M$, $x * y = y * x$

SOME RESULTS

Definition 3.1: Consider that two elements in $BFS(-x_n, x_p)$ and $(-y_n, y_p)$ such that

$0 \leq -(x_n + y_n) \leq -1$ and $0 \leq (x_p + y_p) \leq 1$

i. $[(-x_n, x_p) \oplus (-y_n, y_p)] = [-\{(x_n + y_n) \wedge 1\}, \{(x_p + y_p) \wedge 1\}]$

ii. $[(-x_n, x_p) \odot (-y_n, y_p)] = [-\{(x_n + y_n - 1) \vee 0\}, \{(x_p + y_p - 1) \vee 0\}]$

Definition 3.2: Consider two Bipolar fuzzy matrices $X = [x_{ij}]_{m \times n}$ and $Y = [y_{ij}]_{m \times n}$ where $x = (-x_{ijn}, x_{ijp})$ and $y = (-y_{ijn}, y_{ijp})$ for all i, j. Now define the operator $\oplus$ and $\odot$ on BFMs

$X \oplus Y = [-\{(x_{ijn} + y_{ijn}) \wedge 1\}, \{(x_{ijp} + y_{ijp}) \wedge 1\}]$

$X \odot Y = [-\{(x_{ijn} + y_{ijn} - 1) \vee 0\}, \{(x_{ijp} + y_{ijp} - 1) \vee 0\}]$

Using these operators for any $BFM, X \in F_{mn}$, The authors have proved X is symmetric, Associative , De Morgan's law and Commutative satisfies over $\oplus$ and $\odot$.

Proposition 3.1: If X and Y are symmetric, then $X \oplus Y$ and $X \odot Y$ are symmetric.

Proof: Since X and Y are symmetric

$(-x_{ijn}, x_{ijp}) = (-x_{jin}, x_{jip})$ and $(-y_{ijn}, y_{ijp}) = (-y_{jin}, y_{jip})$

Now let $(-z_{ijn}, z_{ijp})$ and $(-u_{ijn}, u_{ijp})$ be the$i\,j^{th}$ elements of $X \oplus Y$ and $X \odot Y$

To prove $X \oplus Y$ is symmetric

Let $(-z_{ijn}, z_{ijp})$ be the ij^{th} elements of $X \oplus Y$

$$(-z_{ijn}, z_{ijp}) = [-\{(x_{ijn} + y_{ijn}) \wedge 1\}, \{(x_{ijp} + y_{ijp}) \wedge 1\}]$$
$$= [-\{(x_{jin} + y_{jin}) \wedge 1\}, \{(x_{jip} + y_{jip}) \wedge 1\}] = (-z_{jin}, z_{jip})$$

Hence $X \oplus Y$ is symmetric

To prove $X \odot Y$ is symmetric

Let$(-u_{ijn}, u_{ijp})$ be the ij^{th} elements of $X \odot Y$

$$(-u_{ijn}, u_{ijp}) = [-\{(x_{ijn} + y_{ijn} - 1) \vee 0\}, \{(x_{ijp} + y_{ijp} - 1) \vee 0\}]$$

$= [-\{(x_{jin} + y_{jin} - 1) \vee 0\}, \{(x_{jip} + y_{jip} - 1) \vee 0\}]$
$= (-u_{jin}, u_{jip})$

Hence $X \odot Y$ is symmetric.

Proposition 3.2: The operators ⊕ and ⊙ are associative

i) $(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z)$

Proof: Let $X = (-x_{ijn}, x_{ijp})$, $Y = (-y_{ijn}, y_{ijp})$ and $Z = (-z_{ijn}, z_{ijp})$

Let ij^{th} elements of BFMS $X \oplus Y$, $Y \oplus Z$, $(X \oplus Y) \oplus Z$ and $X \oplus (Y \oplus Z)$ are as follows.

$$(-q_{ijn}, q_{ijp}) = [-\{(x_{ijn} + y_{ijn}) \wedge 1\}, \{(x_{ijp} + y_{ijp}) \wedge 1\}]$$
$$(-r_{ijn}, r_{ijp}) = [-\{(y_{ijn} + z_{ijn}) \wedge 1\}, \{(y_{ijp} + z_{ijp}) \wedge 1\}]$$
$$(-s_{ijn}, s_{ijp}) = [-\{(q_{ijn} + z_{ijn}) \wedge 1\}, \{(q_{ijp} + z_{ijp}) \wedge 1\}]$$
$$(-t_{ijn}, t_{ijp}) = [-\{(x_{ijn} + r_{ijn}) \wedge 1\}, \{(x_{ijp} + r_{ijp}) \wedge 1\}]$$

Now we have to prove $(-s_{ijn}, s_{ijp}) = (-t_{ijn}, t_{ijp})$

Case 1: If $(x_{ijn} + y_{ijn}) < 1$ and $(x_{ijp} + y_{ijp}) < 1$ then

$$(-q_{ijn}, q_{ijp}) = [-\{(x_{ijn} + y_{ijn}) \wedge 1\}, \{(x_{ijp} + y_{ijp}) \wedge 1\}]$$
$$= [-(x_{ijn} + y_{ijn}), (x_{ijp} + y_{ijp})]$$
$$(-s_{ijn}, s_{ijp}) = [-\{(x_{ijn} + y_{ijn} + z_{ijn}) \wedge 1\}, \{(x_{ijp} + y_{ijp} + z_{ijp}) \wedge 1\}]$$
$$= [(x_{ijn} + y_{ijn} + z_{ijn}), (x_{ijp} + y_{ijp} + z_{ijp})]$$

Subcase 1.1: If $x_{ijn} + y_{ijn} + z_{ijn} < 1$ Then $s_{ijn} = x_{ijn} + y_{ijn} + z_{ijn}$.

In this case $y_{ijn} + z_{ijn} \leq 1$, then $t_{ijn} = (x_{ijn} + y_{ijn} + z_{ijn}) \wedge 1$

$= x_{ijn} + y_{ijn} + z_{ijn} = s_{ijn}$ since $x_{ijp} + y_{ijp} < 1$ either $y_{ijp} + z_{ijp} < 1$ or > 1

Suppose $y_{ijp} + z_{ijp} < 1$ then $t_{ijp} = (x_{ijp} + y_{ijp} + z_{ijp}) \wedge 1$

$$= x_{ijp} + y_{ijp} + z_{ijp} = s_{ijp}$$

Suppose $y_{ijp} + z_{ijp} > 1$ then $t_{ijp} = (x_{ijp} + y_{ijp} + z_{ijp}) \wedge 1$

$$= x_{ijp} + y_{ijp} + z_{ijp} = s_{ijp}$$

Subcase 1.2: If $x_{ijn} + y_{ijn} + z_{ijn} > 1$ then $(-s_{ijn}, s_{ijp}) = (-1, x_{ijp} + y_{ijp} + z_{ijp})$

In this case $y_{ijn} + z_{ijn} < 1$ or > 1

If $y_{ijn} + z_{ijn} < 1$

$$(-t_{ijn}, t_{ijp}) = [-\{(x_{ijn} + r_{ijn}) \wedge 1\}, x_{ijp} + y_{ijp} + z_{ijp}]$$
$$= [-\{(x_{ijn} + y_{ijn} + z_{ijn}) \wedge 1\}, x_{ijp} + y_{ijp} + z_{ijp}]$$
$$= (-1, x_{ijp} + y_{ijp} + z_{ijp}) = (-s_{ijn}, s_{ijp})$$

If $y_{ijn} + z_{ijn} > 1$

$$(-t_{ijn}, t_{ijp}) = [-\{(x_{ijn} + r_{ijn}) \wedge 1\}, \{(x_{ijp} + r_{ijp}) \wedge 1\}]$$
$$= (-1, x_{ijp} + y_{ijp} + z_{ijp}) = (-s_{ijn}, s_{ijp})$$

Case 2: If $(x_{ijn} + y_{ijn}) \geq 1$ and $(x_{ijp} + y_{ijp}) < 1$

Then $(-s_{ijn}, s_{ijp}) = [-\{(x_{ijn} + y_{ijn} + z_{ijn}) \wedge 1\}, \{(x_{ijp} + y_{ijp} + z_{ijp}) \wedge 1\}]$

$$= (-1, x_{ijp} + y_{ijp} + z_{ijp})$$

Since $(x_{ijn} + y_{ijn}) \geq 1$

$$\Longrightarrow x_{ijn} + y_{ijn} + z_{ijn} \geq 1 \text{ and} x_{ijp} + y_{ijp} < 1$$

$$\Longrightarrow y_{ijn} + z_{ijn} \geq 1 \text{ or } \leq 1 \text{ and } y_{ijp} + z_{ijp} < 1$$

Subcase 2.1: If $y_{ijn} + z_{ijn} \geq 1$ and $y_{ijp} + z_{ijp} < 1$ then

$$\begin{aligned}(-t_{ijn}, t_{ijp}) &= [-\{(x_{ijn} + y_{ijn} + z_{ijn}) \wedge 1\}, (x_{ijp} + y_{ijp} + z_{ijp})] \\ &= (-1, x_{ijp} + y_{ijp} + z_{ijp}) \\ &= (-s_{ijn}, s_{ijp})\end{aligned}$$

Subcase 2.2: If $y_{ijn} + z_{ijn} > 1$ and $y_{ijp} + z_{ijp} > 1$ then

$$\begin{aligned}(-t_{ijn}, t_{ijp}) &= [-\{(x_{ijn} + y_{ijn} + z_{ijn}) \wedge 1\}, \{(x_{ijp} + y_{ijp} + z_{ijp}) \wedge 1\}] \\ &= (-1, x_{ijp} + y_{ijp} + z_{ijp}) \\ &= (-s_{ijn}, s_{ijp})\end{aligned}$$

Subcase 2.3: If $y_{ijn} + z_{ijn} < 1$ and $y_{ijp} + z_{ijp} < 1$ then

$$\begin{aligned}(-t_{ijn}, t_{ijp}) &= (-1, x_{ijp} + y_{ijp} + z_{ijp}) \\ &= (-s_{ijn}, s_{ijp})\end{aligned}$$

Subcase 2.4: If $y_{ijn} + z_{ijn} < 1$ and $y_{ijp} + z_{ijp} > 1$.

This is true from subcase 2.2 and Subcase 2.3.

Case 3: If $x_{ijn} + y_{ijn}$ 1 and $x_{ijn} + y_{ijn} > 1$ then

$$(-s_{ijn}, s_{ijp}) = [-\{(x_{ijn} + y_{ijn} + z_{ijn}) \wedge 1\}, \{(x_{ijp} + y_{ijp} + z_{ijp}) \wedge 1\}]$$

Subcase 3.1: If $x_{ijn} + y_{ijn} + z_{ijn} < 1$ and $x_{ijp} + y_{ijp} + z_{ijp} < 1$

Using subcase 1.1 $s_{ijn} = t_{ijn}$ and $s_{ijp} = x_{ijp} + y_{ijp} + z_{ijp}$ Now $x_{ijp} + y_{ijp} + z_{ijp}$

If $y_{ijp} + z_{ijp} < 1$ then $t_{ijp} = (x_{ijp} + y_{ijp} + z_{ijp}) \wedge = x_{ijp} + y_{ijp} + z_{ijp} = s_{ijp}$

If $y_{ijp} + z_{ijp} > 1$ then $t_{ijp} = x_{ijp} + y_{ijp} + z_{ijp} = s_{ijp}$

Subcase 3.2: If $x_{ijn} + y_{ijn} + z_{ijn} < 1$ and $x_{ijp} + y_{ijp} + z_{ijp} > 1$ then

$s_{ijn} = x_{ijn} + y_{ijn} + z_{ijn}$ from Subcase 1.1, $t_{ijn} = s_{ijn}$

$s_{ijp} = (x_{ijp} + y_{ijp} + z_{ijp}) \wedge 1 = x_{ijp} + y_{ijp} + z_{ijp} = t_{ijp}$

Subcase 3.3: If $x_{ijn} + y_{ijn} + z_{ijn} > 1$ and $x_{ijp} + y_{ijp} + z_{ijp} < 1$ then

From Subcase 1.3 and subcase 3.1 this is true.

Subcase 3.4: If $x_{ijn} + y_{ijn} + z_{ijn} > 1$ and $x_{ijp} + y_{ijp} + z_{ijp} > 1$ then

From Subcase 1.3 and subcase 3.2, $(-s_{ijn}, s_{ijp}) = (-t_{ijn}, t_{ijp})$

Case 4: If $x_{ijn} + y_{ijn} \geq 1$ and $x_{ijp} + y_{ijp} > 1$ then from case 2 and Case 3 it is obvious.

ii) $(\boldsymbol{X} \odot \mathbf{Y}) \odot \boldsymbol{Z} = \boldsymbol{X} \odot (\boldsymbol{Y} \odot \mathbf{Z})$

Proof: Let $(-q_{ijn}, q_{ijp}) = [-\{(x_{ijn} + y_{ijn} - 1) \vee 0\}, \{(x_{ijp} + y_{ijp} - 1) \vee 0\}]$

$$(-r_{ijn}, r_{ijp}) = [-\{(y_{ijn} + z_{ijn} - 1) \vee 0\}, \{(y_{ijp} + z_{ijp} - 1) \vee 0\}]$$

$$(-s_{ijn}, s_{ijp}) = [-\{(q_{ijn} + z_{ijn} - 1) \vee 0\}, \{(q_{ijp} + z_{ijp} - 1) \vee 0\}]$$
$$(-t_{ijn}, t_{ijp}) = [-\{(x_{ijn} + r_{ijn} - 1) \vee 0\}, \{(x_{ijp} + r_{ijp} - 1) \vee 0\}]$$

To prove $(-s_{ijn}, s_{ijp}) = (-t_{ijn}, t_{ijp})$

Case 1: If $(x_{ijn} + y_{ijn} - 1) < 0$ an d $(x_{ijp} y_{ijp} - 1) < 0$ then $(-q_{ijn}, q_{ijp}) = (-0, 0)$

$(-s_{ijn}, s_{ijp}) = [-\{(0 + z_{ijn} - 1) \vee 0\}, \{(0 + z_{ijp} - 1) \vee 0\}] = (0,0)$

Subcase 1.1: If $(x_{ijp} + y_{ijp} + z_{ijp} - 1) < 0$ then $(-s_{ijn}, s_{ijp}) = (0, 0)$ either

$y_{ijp} + z_{ijp} < 0$ and $y_{ijn} + z_{ijn} < 0$

$\Rightarrow (-t_{ijn}, t_{ijp}) = (0,0) = (-s_{ijn}, s_{ijp})$

If $y_{ijn} + z_{ijn} > 0$ and $(y_{ijp} + z_{ijp} - 1) < 0$ then

$(-t_{ijn}, t_{ijp}) = [-\{(x_{ijn} + y_{ijn} + z_{ijn} - 2) \vee 0\}, \{(x_{ijp} + y_{ijp} + z_{ijp} - 2) \vee 0\}]$

Subcase 1.2: If $(x_{ijp} + y_{ijp} + z_{ijp} - 1) > 0$ then

$(-s_{ijn}, s_{ijp}) = (0, x_{ijp} + y_{ijp} + z_{ijp} - 2)$

If $y_{ijn} + z_{ijn} - 1 < 0$ and $y_{ijp} + z_{ijp} - 1 > 0$ then

$(-t_{ijn}, t_{ijp}) = (0, x_{ijp} + y_{ijp} + z_{ijp} - 2) = (-s_{ijn}, s_{ijp})$

If $y_{ijn} + z_{ijn} - 1 < 0$ and $y_{ijp} + z_{ijp} - 1 < 0$ then

$(-t_{ijn}, t_{ijp}) = (0, x_{ijp} + y_{ijp} + z_{ijp} - 2)$

If $y_{ijn} + z_{ijn} - 1 > 0$ and $y_{ijp} + z_{ijp} - 1 > 0$ then

$(-t_{ijn}, t_{ijp}) = [-\{(x_{ijn} + y_{ijn} + z_{ijn} - 2) \vee 0\}, \{(x_{ijp} + y_{ijp} + z_{ijp} - 2) \vee 0\}]$

Case 2: If $x^{n}{}_{ij} + y^{n}{}_{ij} - 1 < 0$ and $x^{p}{}_{ij} + y^{p}{}_{ij} - 1 > 0$ then

$(-s_{ijn}, s_{ijp}) = [-\{(x_{ijn} + y_{ijn} + z_{ijn} - 2) \vee 0\}, \{(x_{ijp} + y_{ijp} + z_{ijp} - 2) \vee 0\}]$

$= (0, x_{ijp} + y_{ijp} + z_{ijp} - 2)$

using subcase 1.1 and subcase 1.2 clearly it is true.

Case 3: If $(x_{ijn} + y_{ijn} - 1) > 0$ and $(x_{ijp} + y_{ijp} - 1) < 0$ then

$(-s_{ijn}, s_{ijp}) = [-\{(x_{ijn} + y_{ijn} + z_{ijn} - 2) \vee 0\}, \{(x_{ijp} + y_{ijp} + z_{ijp} - 2) \vee 0\}]$

$= (-\{(x_{ijn} + y_{ijn} + z_{ijn} - 2), 0\})$

Subcase 3.1: If $x_{ijn} + y_{ijn} + z_{ijn} - 2 < 0$ and $x_{ijp} + y_{ijp} + z_{ijp} - 2 < 0$

then$(-s_{ijn}, s_{ijp}) = 0$

For any values of (y_{ijn}, y_{ijp}), (z_{ijn}, z_{ijp}), $(-s_{ijn}, s_{ijp}) = (-t_{ijn}, t_{ijp})$

Subcase 3.2: If $x_{ijn} + y_{ijn} + z_{ijn} - 2 > 0$ and $x_{ijp} + y_{ijp} + z_{ijp} - 2 < 0$ then

$(-s_{ijn}, s_{ijp}) = (-(x_{ijn} + y_{ijn} + z_{ijn} - 2), 0)$

Since $x_{ijn} + y_{ijn} + z_{ijn} - 2 = (x_{ijn} - 1) + (y_{ijn} + z_{ijn} - 1) >$

$\Rightarrow y_{ijn} + z_{ijn} - 1 > 0$

$(-t_{ijn}, t_{ijp}) = (-(x_{ijn} + y_{ijn} + z_{ijn} - 2), 0) = (-s_{ijn}, s_{ijp})$

Subcase 3.3: If $x_{ijn} + y_{ijn} + z_{ijn} - 2 < 0$ and $x_{ijp} + y_{ijp} + z_{ijp} - 2 > 0$

Searly $s_{ijn} = t_{ijn}$ and $t_{ijp} = x_{ijp} + y_{ijp} + z_{ijp} - 2$

If $y_{ijp} + z_{ijp} > 0 \implies t_{ijp} = x_{ijp} + y_{ijp} + z_{ijp} - 2$

Case 4: $If\ \ x_{ijn}+y_{ijn}-1>0$ and $x_{ijp}+y_{ijp}-1>0$ then

$\left(-s_{ijn},s_{ijp}\right)=\left[-\left\{\left(x_{ijn}+y_{ijn}+z_{ijn}-2\right)\vee 0\right\},\left\{\left(x_{ijp}+y_{ijp}+z_{ijp}-2\right)\vee 0\right\}\right]$

Subcase 4.1: If $y_{ijp}+z_{ijp}-1\geq 0$ then $t_{ijp}=x_{ijp}+y_{ijp}+z_{ijp}-2={s^p}_{ij}$

If $y_{ijp}+z_{ijp}-1<0$ then $t_{ijp}=0=s_{ijp}$

From all the above cases $\left(-s_{ijn},s_{ijp}\right)=\left(-t_{ijn},t_{ijp}\right)$.

Proposition 3.3: Identity

$X\oplus e_b=\left(-x_{ijn},x_{ijp}\right)\oplus(0,0)$

$=\left[-\left\{\left(x_{ijn}+0\right)\wedge 1\right\},\left(x_{ijp}+0\right)\wedge 1\right]$

$=\left[-\left\{x_{ijn}\wedge 1\right\},x_{ijp}\wedge 1\right]$

$=\left(-x_{ijn},x_{ijp}\right)$

Similarly $e_b\oplus X=(0,0)\oplus\left(-x_{ijn},x_{ijp}\right)=\left(-x_{ijn},x_{ijp}\right)$

Hence $X\oplus e_b=e_b\oplus X=X$, where $e_b=(0,0)$ is called the identity element in respect to $\oplus$ operation.

$X\odot e_b=\left(-x_{ijn},x_{ijp}\right)\odot(-1,1)$

$=\left[-\left\{\left(x_{ijn}+1-1\right)\vee 0\right\},\left(x_{ijp}+1-1\right)\vee 0\right]$

$=\left[-\left\{x_{ijn}\vee 0\right\},x_{ijp}\vee 0\right]=\left(-x_{ijn},x_{ijp}\right)$

Similarly $e_b\odot X=(-1,1)\odot\left(-x_{ijn},x_{ijp}\right)=\left(-x_{ijn},x_{ijp}\right)$

Hence $X\odot e_b=e_b\odot X=X$, where $e_b=(-1,1)$ is called the identity element in respect to $\odot$ operation.

Proposition 3.4: Demorgan's Law:

i) $(\boldsymbol{X}\oplus\boldsymbol{Y})^{\boldsymbol{C}}=\boldsymbol{X}^{\boldsymbol{C}}\odot\boldsymbol{Y}^{\boldsymbol{C}}$

ii) $(\boldsymbol{X}\odot\boldsymbol{Y})^{\boldsymbol{C}}=\boldsymbol{X}^{\boldsymbol{C}}\oplus\boldsymbol{Y}^{\boldsymbol{C}}$

Proof. Case 1: If $\left(x_{ijn}+y_{ijn}\right)<1$ and $\left(x_{ijp}+y_{ijp}\right)>1$ then

$(\boldsymbol{X}\oplus\boldsymbol{Y})^{\boldsymbol{C}}=(\left(-x_{ijn},x_{ijp}\right)\oplus\left(-y_{ijn},y_{ijp}\right))^C$

$=\left[-\left\{\left(x_{ijn}+y_{ijn}\right)\wedge 1\right\},\left\{\left(x_{ijp}+y_{ijp}\right)\wedge 1\right\}\right])^C$

$=\left[-\left\{\left(x_{ijn}+y_{ijn}\right)\right\},1\right]^C$

$=\left[-1+\left\{\left(x_{ijn}+y_{ijn}\right)\right\},1-1\right]$

$=\left[-1+\left\{\left(x_{ijn}+y_{ijn}\right)\right\},0\right]$ (1)

$\boldsymbol{X}^{\boldsymbol{C}}\odot\boldsymbol{Y}^{\boldsymbol{C}}=\left(-x_{ijn},x_{ijp}\right)^C\odot\left(-y_{ijn},y_{ijp}\right)^C$

$=\left(-1+x_{ijn},1-x_{ijp}\right)\odot\left(-1+y_{ijn},1-y_{ijp}\right)$

$=\left(-(1-x_{ijn}),1-x_{ijp}\right)\odot\left(-(1-y_{ijn}),1-y_{ijp}\right)$

$=\left[-\left\{\left(1-x_{ijn}+1-y_{ijn}-1\right)\vee 0\right\},\left\{\left(1-x_{ijp}+1-y_{ijp}-1\right)\vee 0\right\}\right]$

$=\left[-\left\{\left(1-(x_{ijn}+y_{ijn})\right)\vee 0\right\},\left\{\left(1-(x_{ijp}+y_{ijp})\right)\vee 0\right\}\right]$

$=\left[-(1-\left\{\left(x_{ijn}+y_{ijn}\right)\right\},0\right]$

$=\left[-1+\left\{\left(x_{ijn}+y_{ijn}\right)\right\},0\right]$ (2)

From (1) & (2), $(\boldsymbol{X}\oplus\boldsymbol{Y})^{\boldsymbol{C}}=\boldsymbol{X}^{\boldsymbol{C}}\oplus\boldsymbol{Y}^{\boldsymbol{C}}$

Case 2: If $\left(x_{ijn} + y^{n}{}_{ij}\right) < 1$ and $\left(x_{ijp} + y_{ijp}\right) < 1$ then

$(\boldsymbol{X} \oplus \boldsymbol{Y})^{\boldsymbol{C}} = ((-x_{ijn}, x_{ijp}) \oplus (-y_{ijn}, y_{ijp}))^{C}$

$= \left[-\{(x_{ijn} + y_{ijn}) \wedge 1\}, \{(x_{ijp} + y_{ijp}) \wedge 1\}\right]^{C}$

$= \left[-(x_{ijn} + y_{ijn}), (x_{ijp} + y_{ijp})\right]^{C}$

$= \left[-1 + (x_{ijn} + y_{ijn}), 1 - (x_{ijp} + y_{ijp})\right]$

$= \left[-1 + (x_{ijn} + y_{ijn}), 1 - (x_{ijp} + y_{ijp})\right]$ (1)

$\boldsymbol{X}^{\boldsymbol{C}} \odot \boldsymbol{Y}^{\boldsymbol{C}} = \left(-x_{ijn}, x_{ijp}\right)^{C} \odot \left(-y_{ijn}, y_{ijp}\right)^{C}$

$= \left(-1 + x_{ijn}, 1 - x_{ijp}\right) \odot \left(-1 + y_{ijn}, 1 - y_{ijp}\right)$

$= \left(-(1 - x_{ijn}), 1 - x_{ijp}\right) \odot \left(-(1 - y_{ijn}), 1 - y_{ijp}\right)$

$= \left[-\{(1 - x_{ijn} + 1 - y_{ijn} - 1) \vee 0\}, \{(1 - x_{ijp} + 1 - y_{ijp} - 1) \vee 0\}\right]$

$= \left[-\left\{\left(1 - (x_{ijn} + y_{ijn})\right) \vee 0\right\}, ((1 - (x_{ijp} + y_{ijp})) \vee 0\right]$

$= \left[-(1 - (x_{ijn} + y_{ijn})), (1 - (x_{ijp} + y_{ijp}))\right]$

$= \left[-1 + (x_{ijn} + y_{ijn}), 1 - (x_{ijp} + y_{ijp})\right]$ (2)

From (1) & (2), $(\boldsymbol{X} \oplus \boldsymbol{Y})^{\boldsymbol{C}} = \boldsymbol{X}^{\boldsymbol{C}} \oplus \boldsymbol{Y}^{\boldsymbol{C}}$

Case 3: If $\left(x_{ijn} + y_{ijn}\right) > 1$ and $\left(x_{ijp} + y_{ijp}\right) > 1$ then $(\boldsymbol{X} \oplus \boldsymbol{Y})^{\boldsymbol{C}} = \boldsymbol{X}^{\boldsymbol{C}} \oplus \boldsymbol{Y}^{\boldsymbol{C}}$=[0,0]

Case 4: If $\left(x_{ijn} + y^{n}{}_{ij}\right) > 1$ and $\left(x_{ijp} + y_{ijp}\right) < 1$ then

$(\boldsymbol{X} \oplus \boldsymbol{Y})^{\boldsymbol{C}} = \boldsymbol{X}^{\boldsymbol{C}} \oplus \boldsymbol{Y}^{\boldsymbol{C}} = \left[0, 1 - (x_{ijp} + y_{ijp})\right]$

To Prove ii) $(\boldsymbol{X} \odot \boldsymbol{Y})^{\boldsymbol{C}} = \boldsymbol{X}^{\boldsymbol{C}} \oplus \boldsymbol{Y}^{\boldsymbol{C}}$

Case 1: If $\left(x_{ijn} + y^{n}{}_{ij}\right) < 1$ and $\left(x_{ijp} + y_{ijp}\right) > 1$ then $(\boldsymbol{X} \odot \boldsymbol{Y})^{\boldsymbol{C}} = \left((-x_{ijn}, x_{ijp}) \odot (-y_{ijn}, y_{ijp})\right)^{C}$

$= \left(\left[-\{(x_{ijn} + y_{ijn} - 1) \vee 0\}, \{(x_{ijp} + y_{ijp} - 1) \vee 0\}\right]\right)^{C}$

$= \left[-\{(x_{ijn} + y_{ijn} - 1) \vee 0\}, \{(x_{ijp} + y_{ijp} - 1) \vee 0\}\right]^{C}$

$= \left[-\left[0, x_{ijp} + y_{ijp} - 1\right]\right]^{C}$

$= \left[-1 + 0, 1 - (x_{ijp} + y_{ijp} - 1)\right]$

$= \left[-1, 2 - (x_{ijp} + y_{ijp})\right]$ (1)

$\boldsymbol{X}^{\boldsymbol{C}} \oplus \boldsymbol{Y}^{\boldsymbol{C}} = \left(-x_{ijn}, x_{ijp}\right)^{C} \oplus \left(-y_{ijn}, y_{ijp}\right)^{C}$

$= \left(-1 + x_{ijn}, 1 - x_{ijp}\right) \oplus \left(-1 + y_{ijn}, 1 - y_{ijp}\right)$

$= \left(-(1 - x_{ijn}), 1 - x_{ijp}\right) \oplus \left(-(1 - y_{ijn}), 1 - y_{ijp}\right)$

$= \left[-\{(1 - (x_{ijn} + 1 + y_{ijn})) \wedge 1\}, \{(1 - x_{ijp} + 1 - +v) \wedge 1\}\right]$

$= \left[-\left(2 - (x_{ijn} + y_{ijn})\right) \wedge 1, (2 - (x_{ijp} + y_{ijp})) \wedge 1\right]$

$= \left[-1, 2 - (x_{ijp} + y_{ijp})\right]$ (2)

From (1) & (2), $(\boldsymbol{X} \odot \boldsymbol{Y})^{\boldsymbol{C}} = \boldsymbol{X}^{\boldsymbol{C}} \oplus \boldsymbol{Y}^{\boldsymbol{C}}$

Case 2: If $\left(x_{ijn} + y_{ijn}\right) < 1$ and $\left(x_{ijp} + y_{ijp}\right) < 1$ then

$(\boldsymbol{X} \odot \boldsymbol{Y})^{\boldsymbol{C}} = ((-x_{ijn}, x_{ijp})\odot(-y_{ijn}, y_{ijp}))^{C}$

$= [-\{(x_{ijn} + y_{ijn} - 1) \vee 0\}, (x_{ijp} + y_{ijp} - 1) \vee 0]^{C}$

$= [0,0]^{C}$

$= [-1 + 0, 1 - 0]$

$= [-1,1]$ (1)

$\boldsymbol{X}^{\boldsymbol{C}} \oplus \boldsymbol{Y}^{\boldsymbol{C}} = (-x_{ijn}, x_{ijp})^{C} \oplus (-y_{ijn}, y_{ijp})^{C}$

$= (-1 + x_{ijn}, 1 - x_{ijp})\oplus(-1 + y_{ijn}, 1 - y_{ijp})$

$= (-(1 - x_{ijn}), 1 - x_{ijp})\oplus(-(1 - y_{ijn}), 1 - y_{ijp})$

$= [-\{(1 - x_{ijn} + 1 - y_{ijn}) \wedge 1\}, (1 - x_{ijp} + 1 - y_{ijp} - 1) \wedge 1]$

$= \left[-\left\{\left(2 - (x_{ijn} + y_{ijn})\right) \wedge 1\right\}, \left(2 - (x_{ijp} + y_{ijp})\right) \wedge 1\right]$

$= [-1,1]$ (2)

From (1) & (2), $(\boldsymbol{X} \odot \boldsymbol{Y})^{\boldsymbol{C}} = \boldsymbol{X}^{\boldsymbol{C}} \oplus \boldsymbol{Y}^{\boldsymbol{C}}$

Case 3: If $(x_{ijn} + y_{ijn}) > 1$ and $(x_{ijp} + y_{ijp}) > 1$ then

$(\boldsymbol{X} \odot \boldsymbol{Y})^{\boldsymbol{C}} = \boldsymbol{X}^{\boldsymbol{C}} \oplus \boldsymbol{Y}^{\boldsymbol{C}} = [(x_{ijn} + y_{ijn}) - 2), (2 - x_{ijp} + y_{ijp})]$

Case 4: If $(x_{ijn} + y_{ijn}) > 1$ and $(x_{ijp} + y_{ijp}) < 1$ then

$(\boldsymbol{X} \odot \boldsymbol{Y})^{\boldsymbol{C}} = \boldsymbol{X}^{\boldsymbol{C}} \oplus \boldsymbol{Y}^{\boldsymbol{C}} = [(x_{ijn} + y_{ijn}) - 2), 1]$

Proposition 3.5:Commutative

Proof: $\boldsymbol{X} \oplus \boldsymbol{Y} = (-x_{ijn}, x_{ijp})\oplus(-y_{ijn}, y_{ijp})$

$= [-\{(x_{ijn} + y_{ijn}) \wedge 1\}, \{(x_{ijp} + y_{ijp}) \wedge 1\}]$

$= [-\{(y_{ijn} + x_{ijn}) \wedge 1\}, \{(y_{ijp} + x_{ijp}) \wedge 1\}]$

$= (-y_{ijn}, y_{ijp})\oplus(-x_{ijn}, x_{ijp})$

$= \boldsymbol{Y}\oplus\boldsymbol{X}$

$\boldsymbol{X} \odot \boldsymbol{Y} = (-x_{ijn}, x_{ijp})\odot(-y_{ijn}, y_{ijp})$

$= [-\{(x_{ijn} + y_{ijn} - 1) \vee 0\}, \{(x_{ijp} + y_{ijp} - 1) \vee 0\}]$

$= [-\{(y_{ijn} + x_{ijn} - 1) \vee 0\}, \{(y_{ijp} + x_{ijp} - 1) \vee 0\}]$

$= (-y_{ijn}, y_{ijp})\odot(-x_{ijn}, x_{ijp})$

$= \boldsymbol{Y}\odot\boldsymbol{X}$

Proposition 3.5: $(F_{mn}, \oplus .\odot)$ is a commutative monoid

Proof: It is clear from the definition 2.10 and definition 3.2 and the proposition 3.2 and 3.5.

CONCLUSION

This paper deals with the operators bounded sum and bounded product of bipolar fuzzy matrices. Some properties are defined and some properties are proved. Thus, the bounded sum and bounded product of bipolar matrices are very useful for further work.

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