The Monophonic Domination Dimension Number of a Graph

[1]J. SUJI PRIYA AND [2]T. MUTHU NESA BEULA

[1] Research Scholar,(Reg. No:20113162092014),

Department of Mathematics,

Women's Christian College, Nagercoil.

email: sujimrthy@gmail.com

[2]Assistant Professor, Department of Mathematics,

Women's Christian College, Nagercoil.

email: tmnbeula@gmail.com

Affiliated to Manonmaniam Sundaranar University, Abishekapatti,
Tirunelveli- 627 012

Abstract

Let G be a connected graph. For $M \subseteq V(G)$, for each $v \epsilon V$(G) the monophonic resolving set is $mr\ (v/M) = \left(d_m(v, v_1), d_m(v, v_2) \ldots d_m(v, v_k)\right)$, where $M = \{v_1, v_2 \ldots . v_k\}$. M is said to be a monophonic resolving set of G, if $mr(v/M) \neq \mathrm{mr}(u/M)$ for every $u, v \in V(G)$, where $u \neq v$. The minimum cardinality of a monophonic resolving set is called the monophonic dimension of G. It is denoted by $mdim(G)$. A set $M \subseteq V(G)$ is said to be a monophonic resolving dominating set of G. If G is both a monophonic resolving set and a dominating set of G.The minimum cardinality of a monophonic resolving dominating set of G is the monophonic resolving domination number of G and is denoted by $\gamma_{mdim}(G)$. Any monophonic resolving set of cardinality $\gamma_{mdim}(G)$ is called a γ_{mdim}- set of G. In this article, the monophonic domination dimension number of some standard graphs are determined.

Keywords: distance, chord, monophonic path, monophonic distance, resolving set, metric dimension, monophonic metric dimension, domination number, monophonic domination metric dimension.

AMS Subject Classification: 05C38, 05C69.

1. Introduction

Let $G = (V, E)$ be a simple undirected connected graph. The *order* and *size of* G are denoted by n and m respectively. For basic graph theoretical terminology, we refer [1]. The length of the shortest $u - v$ path in G is the distance $d(u, v)$ between vertices u and v in a connected graph G. A $u - v$ path with length $d(u, v)$ is referred to as an $u - v$ $geodesic$. For basic graph theoretic terminology, we refer [1]. Let $W = \{w_1, w_2, \ldots, w_k\} \subset V(G)$ be an ordered set and $v \in V(G)$. The representation $r(v/W)$ of v with respect to W is the k-tuple $\left(d(v, w_1), d(v, w_2), \ldots, d(v, w_k)\right)$. Then W is called a *resolving set* if different vertices of G have different representations with respect to W. A resolving set of minimum number of elements is called a *basis* for G and the cardinality of the basis is known as the *metric dimension* of G, represented by $dim(G)$. These concepts were studied in [2].

A path $P's$ chord is an edge that connects two of its non-adjacent vertices. If a path between two vertices u and v in a connected graph G lacks chords, it is referred to as *monophonic path*. The length of the longest $u - v$ monophonic path in G is the monophonic distance $d_m(u, v)$ between $u$and v. These concepts were studied in [3-,11. 13, 15, 21, 22, 24]. In this article, we study a new metric dimension called the *monophonic metric dimension* of a graph. For $M = \{v_1, v_2 \ldots . v_k\} \subset V(G)$ for each $v \epsilon V$ the representation $mr(v/W)$ of v with respect to W is the k-tuple $mr\,(v/M) = \left(d_m(v, v_1), d_m(v, v_2) \ldots d_m(v, v_k)\right)$. M is said to be a monophonic resolving set of G, if $mr(v/M) \neq mr(u/M)$ for every $u, v \in V$, where $u \neq v$. The minimum cardinality of a monophonic resolving set is called the monophonic dimension of G. It is denoted by $mdim(G)$. Any monophonic resolving set of cardinality $mdim(G)$ is called $mdim$-set of G.This concept was introduced and studied in [23]. The dominating set of a graph G is a set S of vertices G such that every vertex not in S is adjacent to a vertex in S. The domination number of G is denoted by $\gamma(G)$ is the minimum size of a dominating set. These concepts were studied in [6, 10, 12, 14, 16, 17, 19, 20]

2. The Monophonic Domination Dimension Number of a Graph

Definition.2.1. Let G be a connected graph. A set $M \subseteq V(G)$ is said to be a monophonic resolving dominating set of G if G is both a monophonic resolving set and a dominating set of G.

The minimum cardinality of a monophonic resolving dominating set of G is the monophonic resolving domination number of G and is denoted by $\gamma_{mdim}(G)$. Any monophonic resolving set of cardinality $\gamma_{mdim}(G)$ is called a γ_{mdim}- set of G.

Example.2.2 For the graph G is given in Figure 1, $M_1 = \{v_3, v_7\}$ is the unique γ-set of G, which is not a resolving set of G and so $\gamma_{mdim}(G) \geq 3$. Let $M_2 = \{v_2, v_3, v_7\}$.Then

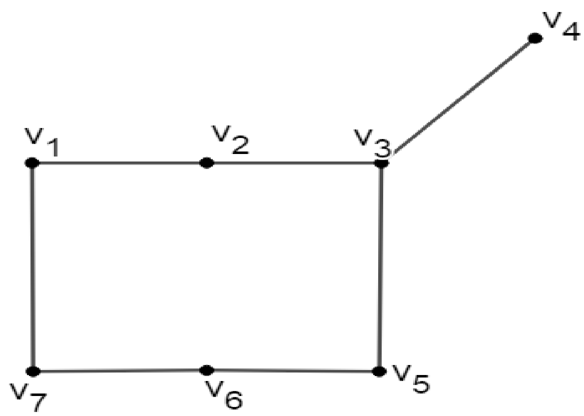

Figure 1

$mr(v_1/M_2) = (1,4,1)$,$mr(v_2/M_2) = (0,1,4)$, $mr(v_3/M_2) = (1,0,3)$,

$mr(v_4/M_2) = (2,1,4)$,…,$mr(v_5/M_2) = (4,1,4)$, $mr(v_6/M_2) = (3,4,1)$,

$mr(v_7/M_2) = (4,3,0)$. Since each representation are distinct, M_2 is a monophonic resolving set of G. Also M_2 is a dominating set of G. Hence M_2 is a monophonic resolving dominating set of G so that $\gamma_{mdim}(G) = 3$. ■

Theorem.2.3. For a star graph $G = K_{1,n-1}$ $(n \geq 3)$. Then $\gamma_{mdim}(G) = n - 1$.

Proof. Let $M = \{v_1, v_2, .. v_{n-2}\}$, Then

$mr(x/M) = (0,1,1, \ldots ,1,1)$, $mr(v_1/M) = (1,0,2, \ldots ,2,2)$,$mr(v_2/M) = (1,2,0, \ldots 2,2)$,

$mr(v_3/M) = (1,2,2,0, \ldots 2,2)$,…,$mr(v_{n-2}/M) = (1,2, \ldots . ,0,2)$, $mr(v_{n-1}/M) = (1,2, \ldots . ,2,2)$.

Since each representation are distinct, M is monophonic resolving set of G. Also M is a dominating set of G. Hence M is a monophonic resolving dominating set of G so that and so $\gamma_{mdim}(G) \leq n-1$. We prove that $\gamma_{mdim}(G) = n-1$ .On the contrary suppose that $\gamma_{mdim}(G) \leq n-2$. Then there exist a γ_{mdim}- set $|M'|$ such that $|M'| \leq n-2$. Then M' is neither a domination set nor a monophonic resolving set of G, which is a contradiction. Therefore $\gamma_{mdim}(G) = n-1$. ■

Theorem.2.4. For the complete bipartite graph $G = K_{r,s}$ $(2 \leq r \leq s)$, $\gamma_{mdim}(G) = r+s-2$.

Proof. Let $X = \{x_1, x_2, \dots, x_r\}$ and $y = \{y_1, y_2, \dots, y_s\}$ be the two bipartite sets of G.

Let $M = \{x_1, x_2, .. x_{r-1}\} \cup \{y_1, y_2, \dots, y_{s-1}\}$. Then

$mr(x_1/M) = (0,\ 2,2, \dots, 2,1,1, \dots, 1)$, $mr(x_2/M) = (2,0,2, \dots, 2,1,1, \dots, 1)$,

$mr(x_3/M) = (2,2,0, \dots, 2,1,1, \dots, 1), \dots$, $mr(x_{r-1}/M) = (\ 2,2, \dots 2,0,1,1, \dots, 1)$,

$mr(x_r/M) = (2,2,2, \dots, 2,1,1, \dots, 1)$, $mr(y_1/M) = (1,1,1, \dots, 0,2,2, \dots, 2)$,

$mr(y_2/M) = (1,1, \dots, 1,2,0,2, \dots, 2)$,$mr(y_3/M) = (1,1, \dots, 1,2,2,0, \dots, 2), \dots$,

$mr(y_{s-1}/M) = (1,1, \dots, 1,2,2, \dots, 0)$, $mr(y_s/M) = (1,1, \dots, 1,2,2, \dots, 2)$.

Since each representation are distinct, M is monophonic resolving set. Since the vertices x_r and y_s are dominated by at least one element of M, M is a dominating set of G.Therefore M is a monophonic resolving dominating set of G and so $\gamma_{mdim}(G) \leq r+s-2$ .We prove that $\gamma_{mdim}(G) = r+s-2$.On the contrary suppose that $\gamma_{mdim}(G) \leq r+s-3$. Then there exists a $\gamma_{mdim}(G)$-set M' such that $|M'| < r+s-3$. Then there exists at least three elements

$u, v, w \in V(G)$ such that $u, v, w \notin M'$.Without loss of generality, let us assume that $u, v \notin X$. Let $u = x_{r-1}$ and $v = x_{r_0}$ .Then $mr(u\ /M') = mr(v/M') = (2,2, \dots, 2,1,1, \dots, 1)$. Which is a contradiction. Therefore $\gamma_{\mathrm{mdim}}(G) = r+s-2$. ■

Theorem.2.5. For a fan graph $G = K_1 + P_{n-1}$ $(n \geq 5)$.Then $\gamma_{mdim}(G) = 2$.

Proof. Let $V(K_1) = x$ and $V(P_{n-1}) = \{v_1, v_2, .., v_{n-1}\}$. Let $M = \{x, v_1\}$.Then

$mr(x/M) = (0,1)$,$mr(v_1/M) = (1,0)$,$mr(v_2/M) = (1,1)$, $mr(v_3/M) = (1,2)$,

$mr(v_4/M) = (1,3)$,…,$mr(v_{n-1}/M) = (1, n-2)$.

Since each representations are distinct, M is a monophonic resolving set of G. Since x is a universal vertices of G and $x \in M$, M is a dominating set of G. Hence M is a monophonic resolving dominating set of G. Therefore $\gamma_{mdim}(G) = 2$. ■

Theorem.2.6. For a Wheel graph $G = K_1 + C_{n-1}$ $(n \geq 5)$. Then $\gamma_{mdim}(G) = \begin{cases} 2 \; for \; n = 5 \\ 3 \; for \; n \geq 6 \end{cases}$

Proof. Let $V(K_1) = x$ and $V(C_{n-1}) = \{v_1, v_2, .., v_{n-1}\}$. For $n = 5$ .Let $M = (v_1, v_2)$ .Then $mr(x/M) = (1,1)$,$mr(v_1/M) = (0,1)$,$mr(v_2/M) = (1,0)$, $mr(v_3/M) = (2,1)$, …,$mr(v_{n-1}/M) = (1,2)$. Since each representations are distinct, M is a monophonic resolving set of G. Since x is a universal vertices of G and $x \in M$, M is a dominating set of G. Hence M is a monophonic resolving dominating set of G. Therefore $\gamma_{mdim}(G) = 2$.

Let $n \geq 6$. It is easily verified that no two element subset of $V(G)$ is not a monophonic resolving dominating set of G, and so $\gamma_{mdim}(G) \geq 3$.

Let $M_1 = \{x, v_1, v_2\}$.Then $mr(x/M_1) = (0,1,1)$,$mr(v_1/M_1) = (1,0,1)$,

$mr(v_2/M_1) = (1,1,0)$, $mr(v_3/M_1) = (1, n-3,1)$,$mr(v_4/M_1) = (1, n-4, n-3)$, $mr(v_5/M_1) = (1, n-3, n-4)$,…,$mr(v_{n-1}/M_1) = (1, 1, n-3)$.

Since each representations are distinct, M_1 is a monophonic resolving set of G. Since x is a universal vertices of G, and $x \in M_1$. M_1 is a monophonic resolving dominating set of G. Hence $M_1$is a monophonic resolving dominating set of G.Therefore $\gamma_{mdim}(G) = 3$. ■

Theorem.2.7 For the path $G = P_n (n \geq 4)$, Then $\gamma_{mdim}(G) = \begin{cases} \frac{n}{3} & \text{if } n \equiv 0 (\text{mod } 3) \\ \left\lceil \frac{n}{3} \right\rceil & \text{Otherwise} \end{cases}$

Proof. Let $V(P_n) = \{v_1, v_2, v_3, \dots, v_n\}$,we have the following cases

Case (i) Let $n \equiv 0 (\text{mod } 3)$. Let $n = 3k, k \geq 3$.Then $M = \{v_2, v_5, \dots, v_{3k-1}\}$ is a dominating set of G,

$mr(v_1/M) = (1,4,7, \dots ,3k-2)$,$mr(v_2/M) = (0,3,6, \dots ,3k-3)$,

$mr(v_3/M) = (1,2,5, \dots ,3k-4)$, $mr(v_4/M) = (2,1,4, \dots ,3k-5)$,

$mr(v_5/M) = (3,0,3, \dots ,3k-6)$, $mr(v_6/M) = (4,1,2, \dots ,3k-7)$,

$mr(v_7/M) = (5,2,1, \dots ,3k-8)$, $mr(v_8/M) = (6,3,0, \dots ,3k-9)$,

$mr(v_9/M) = (7,4,1 \dots ,3k-10)$,…, $mr(v_{3k-4}/M) = (3\text{k}-5,3\text{k}-8,3\text{k}-11, \dots ,1,4)$, $mr(v_{3k-3}/M) = (3\text{k}-4,3\text{k}-7,3\text{k}-10, \dots ,2,5)$, $mr(v_{3k-2}/M) = (3\text{k}-3,3\text{k}-6,3\text{k}-9, \dots ,3,6)$,$mr(v_{3k-1}/M) = (3\text{k}-2,3\text{k}-5,3\text{k}-8, \dots ,4,7)$.

Since each representations are distinct, M is a monophonic resolving set of G. Also M is a dominating set of G. Hence M is a monophonic resolving dominating set of G and so $\gamma_{mdim}(G) \le \frac{n}{3}$. We prove that $\gamma_{mdim}(G) = \frac{n}{3}$.On the contrary suppose that $\gamma_{mdim}(G) < \frac{n}{3} - 1$. Then there exist a γ_{mdim}-set M' such that $|M'| < \frac{n}{3} - 1$.Then M' is not a dominating set of G, which is a contradiction. Therefore $\gamma_{mdim}(G) = \frac{n}{3}$.

Case(ii) Let $n \equiv 1(\text{mod } 3)$. Let $n = 3k+1, k \ge 3$.Then $M = \{v_2, v_5, v_8, v_{10} \dots , v_{3k}\}$

$mr(v_1/M) = (1,4,7,9 \dots ,3\text{k}-2,3\text{k})$,$mr(v_2/M) = (0,3,6,8, \dots ,3k-3,3k-1)$,

$mr(v_3/M) = (1,2,5,7 \dots ,3k-4,3k-2)$, $mr(v_4/M) = (2,1,4,6 \dots ,3k-5,3k-3)$,

$mr(v_5/M) = (3,0,3,5, \dots ,3k-6,3k-4)$,…, $mr(v_{3k-4}/M) = (3\text{k}-5,3\text{k}-7,3\text{k}-8,3\text{k}-5, \dots 4,2,1,4)$,$mr(v_{3k-3}/M) = (3\text{k}-6,3\text{k}-8,3\text{k}-7,3\text{k}-4, \dots ,5,2,1,3)$,

$mr(v_{3k-2}/M) = (3\text{k}-7,3\text{k}-9,3\text{k}-6,3\text{k}-3, \dots ,2,0,3,6)$,$mr(v_{3k-1}/M) = (3\text{k}-8,3\text{k}-10,3\text{k}-5,3\text{k}-2, \dots 1,1,4,7)$, $mr(v_{3k}/M) = (3\text{k}-9,3k-11,3k-4,3k-1, \dots 0,2,5,8)$

Since each representations are distinct, M is a monophonic resolving set of G. Also M is a dominating set of G. Hence M is a monophonic resolving dominating set of G so that $\gamma_{mdim}(G) \le \left\lceil \frac{n}{3} \right\rceil$.We prove that $\gamma_{mdim}(G) = \left\lceil \frac{n}{3} \right\rceil$ .On the contrary suppose that

$\gamma_{mdim}(G) < \left\lceil \frac{n}{3} \right\rceil - 1$.Then there exist a γ_{mdim}-set M' such that $|M'| < \left\lceil \frac{n}{3} \right\rceil - 1$.Then M' is not a dominating set of G,which is a contradiction. Therefore $\gamma_{mdim}(G) = \left\lceil \frac{n}{3} \right\rceil$

Case(iii) Let $n \equiv 2(\text{mod } 3)$. Let $n = 3k+2, k \geq 1$.Then $M = \{v_2, v_5, v_8, v_{11} \dots, v_{3k-1}\}$

$mr(v_1/M) = (1,4,7,10 \dots ,3k-2)$,$mr(v_2/M) = (0,3,6,9, \dots ,3k-3)$,

$mr(v_3/M) = (1,2,5,8 \dots ,3k-4)$, $mr(v_4/M) = (2,1,4,7, \dots ,3k-5)$,

$mr(v_5/M) = (3,0,3,6, \dots ,3k-6)$, $mr(v_6/M) = (4,1,2,5 \dots ,3k-7)$,

$mr(v_7/M) = (5,2,1,4 \dots ,3k-8)$, $mr(v_8/M) = (6,3,0,3 \dots ,3k-9)$,

$mr(v_9/M) = (7,4,1,2, \dots ,3k-10)$, $mr(v_{10}/M) = (8,5,2,1, \dots ,3k-11)$,

$mr(v_{11}/M) = (9,6,3,0, \dots ,3k-12)$,....,

$mr(v_{3k-5}/M) = (3k-4,3k-7,3k-8,3k-5, \dots 4,2,1,4)$,

$mr(v_{3k-4}/M) = (3k-3,3k-6,3k-9,3k-6, \dots 3,0,3,6)$,

$mr(v_{3k-3}/M) = (3k-2,3k-5,3k-10,3k-7, \dots ,2,1,4,7)$,

$mr(v_{3k-2}/M) = (3k-1,3k-4,3k-11,3k-8, \dots ,1,2,5,8)$,

$mr(v_{3k-1}/M) = (3k, 3k-3,3k-12,3k-9, \dots ,0,3,6,9)$,

$mr(v_{3k}/M) = (3k-9,3k-11,3k-4,3k-1, \dots 0,2,5,8)$.

Since each representations are distinct, M is a monophonic resolving set of G. Also M is a dominating set of G. Hence M is a monophonic resolving dominating set of G so that $\gamma_{mdim}(G) \leq \left\lceil \frac{n}{3} \right\rceil$.We prove that $\gamma_{mdim}(G) = \left\lceil \frac{n}{3} \right\rceil$ .On the contrary suppose that

$\gamma_{\mathrm{mdim}}(G) < \left\lceil \frac{n}{3} \right\rceil - 1$. Then there exist a γ_{mdim}-set M' such that $|M'| < \left\lceil \frac{n}{3} \right\rceil - 1$.Then M' is not a dominating set of G,which is a contradiction. Therefore $\gamma_{mdim}(G) = \left\lceil \frac{n}{3} \right\rceil$. ■

Theorem.2.8 For the cycle graph $G = C_n (n \geq 3)$, Then $\gamma_{mdim}(G) \begin{cases} 2 & \text{if } n \in \{3,4,5\} \\ \frac{n}{3} & \text{if } n \equiv 0 (\text{mod } 3) \\ \left\lceil \frac{n}{3} \right\rceil & \text{Otherwise} \end{cases}$.

Proof. Let C_n be $\{v_1, v_2, v_3, \ldots, v_n, v_1\}$ if $n = 3$.

If $n = 4$, then $M = \{v_1, v_2, v_3\}$ is a γ_{mdim}- set of G so that $\gamma_{\text{mdim}}(G) = 2$.

If $n = 5$, then $M = \{v_1, v_3, v_4\}$ is a γ_{mdim}-set of G so that $\gamma_{mdim}(G) = 2$.

For $n \geq 6$, we consider three cases

Case (i) Let $n \equiv 0 (\text{mod } 3)$. $n = 3k, k \geq 3$. Let $M_1 = \{v_1, v_4, \ldots, v_{3k-2}, v_{3k}\}$ .Then M_1 is a dominating set of G. To prove M_1 is a monophonic resolving set of G.

$mr(\text{v}_1/M_1) = (0,6,6, \ldots, 3\text{k} - 3), mr(\text{v}_2/M_1) = (1,7,5, \ldots, 3k - 4)$,

$mr(v_3/M_1) = (7,1,5, \ldots, 3k - 4), mr(v_4/M_1) = (6,0,6, \ldots, 3k - 3)$,

$mr(v_5/M_1) = (5,1,7, \ldots, 3k - 2), \ldots, mr(v_{3k-3}/M_1) = (3\text{k} - 8, 3\text{k} - 2, 3k - 4, \ldots 1,7,5)$,

$mr(v_{3k-2}/M_1) = (3k - 9, 3k - 3, 3k - 3, \ldots 0,6,6)$,

$mr(v_{3k-1}/M_1) = (3\text{k} - 8, 3k - 4, 3\text{k} - 2, \ldots 1,5,7)$,

$mr(v_{3k}/M_1) = (3k - 2, 3k - 4, 3\text{k} - 8, \ldots, 7,5,1)$.

Since each representations are distinct, M_1 is a monophonic resolving set of G. Hence $M_1$is a monophonic resolving dominating set of G and so $\gamma_{mdim}(G) \leq \frac{n}{3}$. We prove that $\gamma_{mdim}(G) = \frac{n}{3}$ .On the contrary suppose that $\gamma_{\text{mdim}}(G) < \frac{n}{3} - 1$. Then there exist a γ_{mdim}-set $M_1{}'$ such that

$|M_1{}'| < \frac{n}{3} - 1$.Then M_1' is not a dominating set of G, which is a contradiction. Therefore

$\gamma_{mdim}(G) = \frac{n}{3}$.

Case(ii) Let $n \equiv 1 (\text{mod } 3)$. Let $n = 3k + 1, k \geq 3$.Then $M_2 = \{v_1, v_4, \ldots, v_{3k-2}, v_{3k+1}\}$.Then M_2 is a dominating set of G.To prove M_2 is a monophonic resolving set of G. Then

$mr(v_1/M_2) = (0,7,6,1, \dots ,3k-8), mr(v_2/M_2) = (1,8,5,8, \dots ,3k-1),$

$mr(v_3/M_2) = (8,1,6,7 \dots ,3k-2), mr(v_4/M_2) = (7,0,7,6, \dots ,3k-3),$

$mr(v_5/M_2) = (6,1,8,5, \dots ,3k-4), \dots , mr(v_{3k-3}/M_2) = (3k-3,3k-8,3k-1,3k-4, \dots 6,1,8,5), mr(v_{3k-2}/M_2) = (3k-2,3k-9,3k-2,3k-3, \dots ,7,0,7,6),$

$mr(v_{3k-1}/M_2) = (3\text{k}-1,3\text{k}-8,3\text{k}-3,3\text{k}-2, \dots 8,1,6,7), mr(v_{3k}/M_2) = (3\text{k}-8,3\text{k}-1,3\text{k}-4,3\text{k}-1, \dots 1,8,5,8), mr(v_{3k+1}/M_2) = (3\text{k}-9,3\text{k}-2,3\text{k}-3,3\text{k}-8, \dots ,0,7,6,1).$

Since each representations are distinct, M_2 is a monophonic resolving set of G. Also M is a dominating set of G.Hence M_2 is a monophonic resolving dominating set of G.Therefore $\gamma_{mdim}(G) \leq \left\lceil \frac{n}{3} \right\rceil$. We prove that $\gamma_{mdim}(G) = \left\lceil \frac{n}{3} \right\rceil$.On the contrary suppose that $\gamma_{mdim}(G) < \left\lceil \frac{n}{3} \right\rceil - 1$.Then there exist a $\gamma_{mdim}(G)$-set $M_2{}'$ such that $|M_2{}'| < \left\lceil \frac{n}{3} \right\rceil - 1$. Then M' is not a dominating set of G, which is a contradiction. Therefore $\gamma_{mdim}(G) = \left\lceil \frac{n}{3} \right\rceil$.

Case (iii) Let $n \equiv 2 (\text{mod } 3)$. Let $n = 3k+2, k \geq 3$.Then $M_3 = \{v_1, v_4, \dots , v_{3k-2}, v_{3k+1}\}$.Then M_3 is a dominating set of G.To prove M_2 is a monophonic resolving set of G. Then

$mr(v_1/M_3) = (0,8,6,9, \dots ,3k-3,3k), mr(v_2/M_3) = (1,9,6,8, \dots ,3k-3,3k-1),$

$mr(v_3/M_3) = (9,1,7,7, \dots ,3k-2,3k-2), mr(v_4/M_3) = (8,0,8,6, \dots ,3k-1,3k-3),$

$mr(v_5/M_3) = (7,1,9,6, \dots ,3k, 3k-3), \dots, mr(v_{3k-3}/M_3) = (3k-3,3k-8,3k-1,3k-4, \dots 6,1,8,5), mr(v_{3k-2}/M_3) = (3k-1,0,3k-1,3k-3, \dots ,8,0,8,6),$

$mr(v_{3k-1}/M_3) = (3k, 1,3k-2,3k-2, \dots 9,1,7,7), mr(v_{3k}/M_3) = (1,3k, 3k-3,3k-1,, \dots 1,9,6,8), mr(v_{3k+1}/M_3) = (0,3k-1,3k-3,3k, \dots ,0,8,6,9).$

$mr(v_{3k+2}/M_3) = (1,3k-2,3k-2,1, \dots ,1,7,7,1).$

Since each representations are distinct, M_2 is a monophonic resolving set of G.Also M is a dominating set of G.Hence M_2 is a monophonic resolving dominating set of G.Therefore $\gamma_{mdim}(G) \leq \left\lceil \frac{n}{3} \right\rceil$. We prove that $\gamma_{mdim}(G) = \left\lceil \frac{n}{3} \right\rceil$.On the contrary suppose that $\gamma_{mdim}(G) <$

$\left\lceil \frac{n}{3} \right\rceil - 1$.Then there exist a $\gamma_{mdim}(G)$-set $M_3{}'$ such that $|M_3{}'| < \left\lceil \frac{n}{3} \right\rceil - 1$. Then M' is not a dominating set of G, which is a contradiction. Therefore $\gamma_{mdim}(G) = \left\lceil \frac{n}{3} \right\rceil$. ■

Theorem.2.9. For the Alternate triangular cycle graph $G = A(C_{2n})$, $\gamma_{mdim}(G) = 3$.

Proof. An alternate triangular cycle $A(C_{2n})$ is obtained from even cycle $C_{2n} = \{v_1, u_1, v_2, u_2, \ldots, v_n, u_n\}$ by joining v_i and u_i to a new vertex w_i.That is every alternate edge of a cycle is replaced by C_3. Let $n = 4, M = \{v_1, v_2, v_4\}$.Then $mr(v_1/M) = (0,1,1)$,

$mr(v_2/M) = (1,0,n-2), mr(v_3/M) = (n-2,1,1), mr(v_4/M) = (1,n-2,0)$,

$mr(u_1/M) = (1,1,n-1)$,$mr(u_2/M) = (n-1,n-1,1)$. Since each representations are distinct,M is a monophonic resolving set of G. Also M is a dominating set of G. Hence M is a monophonic resolving dominating set of $G$and so $\gamma_{mdim}(G) \leq 3$.We have to prove that $\gamma_{mdim}(G) = 3$ .Suppose that $\gamma_{mdim}(G) \leq 2$.Then there exist a γ_{mdim}-set M' such that

$|M'| < 2$.Then M' is not a dominating set of G.Which is a contradiction. There fore

$\gamma_{mdim}(G) = 3$.

Let $n \geq 6$ and let $M_1 = \{u_1, u_2, u_3 \ldots u_n\}$, Then

$mr(v_1/M_1) = (1, \text{n}-2, \text{n}-1, \ldots, n-1)$,$mr(v_2/M_1) = (1, n-1, n-2, \ldots, n-2)$,

$mr(v_3/M_1) = (n-1,1,n-2, \ldots, n-2), mr(v_4/M_1) = (n-2,1,n-1, \ldots, n-1)$,...,

$mr(v_{n-1}/M_1) = (n-2, n-1,1, \ldots,1), mr(v_n/M_1) = (n-1, n-2,1, \ldots,1)$,

$mr(u_1/M_1) = (0, n-1, n-1, \ldots, n-1), mr(u_2/M_1) = (n-1,0, n-1, \ldots, n-1)$,

$mr(u_3/M_1) = (n-1, n-1,0, \ldots,0)$,... , $mr(u_{n-1}/M_1) = (n-1,0, n-1, \ldots, n-1)$,

$mr(u_n/M_1) = (n-1, n-1,0, \ldots,0)$.

Since each representations are distinct,M_1 is a monophonic resolving set of G. Also M_1 is a dominating set of G. Hence M_1 is a monophonic resolving dominating set of G and so $\gamma_{\text{mdim}}(G) \leq n$ .We have to prove that $\gamma_{mdim}(G) = n$. On the contrary suppose that

$\gamma_{mdim}(G) \leq n-1$.Then there exist a γ_{mdim}-set M_1' such that $|M_1{}'| < n-1$.Then M_1' is not a dominating set of G, which is a contradiction. Therefore $\gamma_{mdim}(G) = n$.

■

Theorem.2.10 For the double wheel graph $G = DW_n, (n \geq 4), \gamma_{mdim}(G) = 3$.

Proof. A double wheel graph DW_n of size n can be composed of $2C_n + K_1$. It consists of two cycles of size n where vertices of two cycles are all connected to a central vertex.

Let $M = \{x, v_1, v_2, u_1, u_2\}, n \geq 4$.Then

$mr(x/M) = (0,1,1,1,1), mr(v_1/M) = (1,0,1, n-2, n-2),$

$mr(v_2/M) = (1,1,0, n-2, n-2), mr(v_3/M) = (1, n-2,1, n-2, n-2), \ldots,$

$mr(v_{n-1}/M) = (1, n-2,1, n-2, n-2), mr(v_n/M) = (1,1, n-2, n-2, n-2),$

$mr(u_1/M) = (1, n-2, n-2,0,1), mr(u_2/M) = (1, n-2, n-2,1,0),$

$mr(u_3/M) = (1, n-2, n-2, n-2,1),\ldots, mr(u_{n-1}/M) = (1, n-2, n-2, n-2,1),$

$mr(u_n/M) = (1, n-2, n-2,1, n-2).$

Since each representations are distinct,M is a monophonic resolving set of G. Also M is a dominating set of G. Hence M is a monophonic resolving dominating set of $G$and so $\gamma_{mdim}(G) \leq 5$.We have to prove that $\gamma_{mdim}(G) = 5$.On the contrary suppose that

$\gamma_{mdim}(G) = 4$.Then there exist a γ_{mdim}-set M' such that $|M'| < 4$.Then M' is a monophonic resolving set, but not a dominating set, which is a contradiction. Next assume that $x \in M'$.Then there exist two distinct vertices $y, z \in V(C_{n-1})$, such that $mr(u_3/M) = mr(u_4/M)$. Which is a contradiction. Therefore $\gamma_{mdim}(G) = 5$. ■

Theorem.2.11. For the crown graph $G = H_{n,n}$, $(n \geq 5)$. Then $\gamma_{mdim}(G) = r + s - 2$.

Proof. An undirected graph with $2n$ vertices in the two sets $\{v_1, v_2, \ldots, v_r\}$ and $\{u_1, u_2, \ldots, u_s\}$ and with an edge from u_i to v_j, whenever $i \neq j$.

Let $X = \{v_1, v_2, \ldots, v_r\}$ and $Y = \{u_1, u_2, \ldots, u_s\}$ be the two bipartite sets of G.

Let $M = \{v_1, v_2 \dots, v_{r-1}, u_1, u_2 \dots, u_{s-1}\}$. Then $mr(v_1/M) = (0,2,\dots,3,1,1,\dots,1)$,

$mr(v_2/M) = (2,0,2,\dots,1,1,\dots,1)$,$mr(v_3/M) = (2,2,0,\dots,2,1,1,\dots,1)$,

$mr(v_4/M) = (2,2,2,0,\dots,2,1,1,\dots,1)$,…,$mr(v_{r-1}/M) = (2,2,\dots 2,0,1,1,\dots,1)$,

$mr(v_r/M) = (2,2,2,\dots,2,1,1,\dots,1)$,$mr(u_1/M) = (3,1,1,\dots 1,0,2,2,\dots,2)$,

$mr(u_2/M) = (1,1,\dots,1,2,0,\dots,2)$,$mr(u_3/M) = (1,1,\dots,1,2,2,0,\dots,2)$,

$mr(u_4/M) = (1,1,\dots,1,2,2,2,0,\dots,2)$,…,$mr(u_{s-1}/M) = (1,1,\dots,1,2,2,\dots 2,0)$,

$mr(u_s/M) = (1,1,\dots,1,2,2,\dots,2)$.

Hence it follows that M is a monophonic resolving set of G. Since the vertices v_r and u_s are dominated by at least one element of M, M is a dominating set of G.Therefore M is a monophonic resolving dominating set of G,and so $\gamma_{mdim}(G) \le r + s - 2$.We prove that $\gamma_{mdim}(G) = r + s - 2$.On the contrary suppose that $\gamma_{mdim}(G) \le r + s - 3$. Then there exist a $\gamma_{mdim}(G)$-set M' such that $|M'| < r + s - 3$. If $M' \subset X \; or \; Y$. Then M' is not a dominating set of G.Therefore $M' \subseteq X \cup Y$.Then there exists $u \in X$ and $v \in Y$ with $u, v \in M'$, such that $mr(u / M') = mr(v / M')$, which is a contradiction. Therefore $\gamma_{mdim}(G) = r + s - 2$.

■

References

[1] F. Buckley, and F. Harary, *Distance in graphs*, Addison-Wesley, Redwood City, CA, (1990).

[2] G. Chartrand, L. Eroh, M.A. Johnson and O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph, Discrete Appl. Math. 105 (1-3), (2000),99-113.

[3] M. C. Dourado, F. Protti, and J. L. Szwarcfiter, Complexity results related to monophonic convexity, *Discrete Applied Mathematics*, 158, (2010), 1268 - 1274.

[4] J. John and S. Panchali, The upper monophonic number of a graph, Int. J. Math. Combin. 4, (2010), 46 – 52.

[5] J. John and S. Panchali, The forcing monophonic number of a graph, International Journal of

Mathemametical Archive, (3)(3),(2012), 935-938.

[6] J. John and P.Arul Paul Sudhahar, The monophonic domination number of a graph, Proceeding of the International conference on Mathematics and Business Management, 1, (2012), 142-145.

[7] J. John and P. Arul Paul Sudhahar and A. Vijayan, The connected edge monophonic number of a graph, 3(2), (2012), 132-136.

[8] J. John,P.Arul Paul Sudhahar, and A.Vijayan, The connected monophonic number of a graph, International Journal of Combinatorial graph theory and applications.5(1), (2012), 41-48.

[9] J. John and P. Arul Paul Sudhahar, The upper connected monophonic number and forcing connected monophonic number of a graph, International Journal of Mathematics Trends and Technology 3 (1), (2012), 29-33.

[10]] J. John, G. Edwin and P. Arul Paul Sudhahar, The Steiner domination number of a graph, International Journal of Mathematics and Computer Applications Research, 3(3), (2013), 37 - 42.

[11] J. John and P. Arul Paul Sudhahar, The upper edge –to-vertex monophonic number of a graph, International Journal of Mathematics and Computer Applications Research 3 (1),(2013), 291-296.

[12] J. John and N. Arianayagam, The detour domination number of a graph, Discrete Mathematics Algorithms and Applications, 09, 01,(2017), 1750006.

[13] J John and K. Uma Samundesvari, Total and forcing total edge-to-vertex monophonic number of a graph, Journal of Combinatorial Optimization 35, (2018), 134-147.

[14] J. John, P. Arul Paul Sudhahar, and D. Stalin, On the (M.D) Number of a graph Proyecciones Journal of Mathematics ,38(2), (2019), 255-266.

[15] J.John, The forcing monophonic and the forcing geodetic numbers of a graph, Indonesian Journal of Combinatorics .4(2) (2020) 114-125.

[16] J. John, and V. Sujin Flower, The edge-to-edge geodetic domination number of a graph, Proyecciones Journal of Mathematics, 40(3), (2021), 635-658.

[17] J. John and M.S. Malchijah, The forcing non-split domination number of a graph, Korean journal of mathematics, 29(1), (2021), 1-12.

[18] J. John, On the vertex monophonic, vertex geodetic and vertex Steiner numbers of graphs, Asian-European Journal of Mathematics 14 (10), (2021), 2150171

[19] J.John, and V. Sujin Flower, On the forcing domination and the forcing total domination numbers of a graph, Graphs and Combinatorics,38, (2022)142.

[20] S.Kavitha, S.Chellathurai and J.John, On the forcing fonnected domination number of a graph, Journal of Discrete Mathematical Sciences and Cryptography, 3(20) (2017) 611-624.

[21] E. M. Paluga, and S. R. Canoy Jr., Monophonic numbers of the join and Composition of connected graphs, Discrete Mathematics, 307, (2007), 1146 –1154.

[22] A.P.Santhakumaran P.Titus, Monophonic distance in Graphs, Discrete Mathematics Algorithms and Applications (3) ,(2011),159.

[23] J.Suji Priya and. T. Muthu Nesa Beula, The monophonic metric dimension of some graphs under join operation. (communicated)

[24] K. Uma Samundesvari and J. John, The edge fixing edge-to-vertex monophonic number of a graph, Applied Mathematics E-Notes 15, (2015), 261-275.