Fixed point theorems on B_4 - metric spaces

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Abstract

Recently, B_4 - metric spaces are introduced as a generalization of metric spaces. In this paper, we introduce new contractive mappings on B_4 - metric spaces and investigate relationships among them. Examples and counterexamples are provided as and when necessary. In addition, we obtain fixed point theorems for self-maps on B_4-metric spaces

Keywords: Diameter, Fixed point theorem, Periodic point, S-metric space, B_4 - metric space.

1. Introduction

The generalizations of the contraction principle in different directions as well as many new fixed point results with applications have been established by different researchers ([1]-[6],[8],[9],[12], [14] - [16]. Recently, K.K.M. Sarma, Ch. Srinivasa Rao, and S. Ravi Kumar [11] have introduced the concept of a B_4 - metric space as a generalization of S - metric space. In this paper, we present results on fixed point theory in B_4 - metric.

2. Preliminaries

Definition 2.1. [11] Let $Y_0 \neq \phi$ and $B_4 : Y_0^4 \to \mathbb{R}$ be a function meeting the criteria below:

$\forall f_0, g_0, h_0, t_0, a \in Y_0$.

$B_4(f_0, g_0, h_0, t_0) \leq$

$B_4(f_0, f_0, f_0, a_0) + B_4(g_0, g_0, g_0, a_0) + B_4(h_0, h_0, h_0, a_0) + B_4(t_0, t_0, t_0, a_0)$. Then, B_4 is called a B_4 - metric on Y_0 and the pair (Y_0, B_4) is called a B_4 - metric space.

Example 2.1. [11] Let $Y_0 \neq \phi$ and define the function $B_4 : Y_0^4 \to \mathbb{R}$

as $B_4(f_0, g_0, h_0, t_0) = \begin{cases} 0, & \text{if } f_0 = g_0 = h_0 = t_0 \\ 1, & \text{otherwise} \end{cases}$

Then, B_4 is a B_4 - metric on Y_0.

Example 2.2. [11] Let $Y_0 = \mathbb{R}$ and define the function $B_4 : Y_0^4 \to \mathbb{R}$ by

$B_4(f_0, g_0, h_0, t_0) = |f_0 - g_0| + |g_0 - h_0| + |f_0 + g_0 + h_0 - 3t_0|, \forall f_0, g_0, h_0, t_0 \in \mathbb{R}$. Then, B_4 is a B_4 - metric on Y_0.

Definition 2.2. [13] Let $Y_0 \neq \phi$, and $S: Y_0^3 \to [0, \infty)$ be a function meeting the criteria below:

$\forall f_0, g_0, h_0, a \in Y_0 S(f_0, g_0, h_0) = 0 \Leftrightarrow f_0 = g_0 = h_0$

$S(f_0, g_0, h_0) \leq S(f_0, f_0, a_0) + S(g_0, g_0, a_0) + S(h_0, h_0, a_0)$. Then, S is called an S - metric on Y_0

and the pair (Y_0, S) is called an S-metric space.

Lemma 2.1. [11] Let (Y_0, B_4) be a B_4 - metric space.Then,

$B_4(f_0, f_0, f_0, g_0) = B_4(g_0, g_0, g_0, f_0)$ (2.1)

The relation between a S - metric and a

B_4 - metric is given in [11] as follows:

Lemma 2.2. [11] Let (Y_0, S) be a S - metric space.

Define $B_{4S}: Y_0^4 \to [0, \infty)$ as follows:

$B_{4S}(f_0, g_0, h_0, t_0) = S(f_0, g_0, h_0) + S(f_0, g_0, t_0) + S(f_0, h_0, t_0) + S(g_0, h_0, t_0)$,

$\forall f_0, g_0, h_0, t_0 \in Y_0$.

Then B_{4S} is a B_4 - metric on X. This B_{4S} - metric is called the B_4 - metric induced by S on Y_0.

Lemma 2.4. [11] Let (Y_0, B_4) be a B_4 - metric space. Define $S_b: Y_0^3 \to \mathbb{R}$ as follows:

$S_b(f_0, g_0, h_0) = B_4(f_0, f_0, g_0, h_0) + B_4(f_0, g_0, g_0, h_0) + B_4(f_0, g_0, h_0, h_0), \forall f_0, g_0, h_0 \in Y_0$.

Then (Y_0, S_b) is a S-metric space with index 2.(S_b is called the S-metric with index 2 induced by B_4).

Example 2.3. Let $Y_0 = \{1,2,3,4\}$ and the function

$B_4: Y_0 \times Y_0 \times Y_0 \times Y_0 \to [0, \infty)$, be defined as

$\mathrm{B}_4(1,1,1,2) = \mathrm{B}_4(2,2,2,1) = \mathrm{B}_4(2,2,2,3) = \mathrm{B}_4(3,3,3,2) = \mathrm{B}_4(3,3,3,4) = \mathrm{B}_4(4,4,4,3) = 3$

$\mathrm{B}_4(1,1,1,3) = \mathrm{B}_4(3,3,3,1) = \mathrm{B}_4(4,4,4,2) = \mathrm{B}_4(2,2,2,4) = 6$, $\mathrm{B}_4(1,1,1,4) = \mathrm{B}_4(4,1,1,1) = 9$,

$\mathrm{B}_4(f_0, g_0, h_0, t_0) = 0$, if $f_0 = g_0 = h_0 = t_0$

$\mathrm{B}_4(f_0, g_0, h_0, t_0) = 1$, otherwise, $\forall f_0, g_0, h_0, t_0 \in Y_0$.

Then B_4 is a B_4 - metric on Y_0.

Example 2.4. Suppose Y_0 is a set with at least 4 elements.Define $B_4: Y_0^4 \to \mathbb{R}^+$by

$$B_4(f_0, g_0, h_0, t_0) = \begin{cases} 0, & \text{if } f_0 = g_0 = h_0 = t_0 \\ 6, & \text{if } f_0, g_0, h_0, t_0 \text{ are distinct} \\ 1, & \text{otherwise} \end{cases}$$

Then, B_4 is B_4 - metric with index 2.

3.Main Results

Now we state various contraction conditions [13].

Definition 3.1. [13] Let (Y_0, S) be an S - metric space. Suppose T_0 is a self-map on Y_0. Define various contraction conditions on Y_0 as follows: for each $f_0, g_0 \in Y_0, f_0 \neq g_0$. (3.1)

$$S(T_0 f_0, T_0 f_0, T_0 g_0) < \max \begin{Bmatrix} S(f_0, f_0, g_0), \\ S(T_0 f_0, T_0 f_0, f_0), \\ S(T_0 g_0, T_0 g_0, g_0), \\ S(T_0 g_0, T_0 g_0, f_0), \\ S(T_0 f_0, T_0 f_0, g_0) \end{Bmatrix}$$

There exists a positive integer p such that(3.2)

$$S(T_0^p f_0, T_0^p f_0, T_0^p g_0) < \max \begin{Bmatrix} S(f_0, f_0, g_0), \\ S(T_0^p f_0, T_0^p f_0, f_0), \\ S(T_0^p g_0, T_0^p g_0, g_0), \\ S(T_0^p g_0, T_0^p g_0, f_0), \\ S(T_0^p f_0, T_0^p f_0, g_0) \end{Bmatrix}$$

There exist positive integers p and q such that,

(3.3) $S(T_0^p f_0, T_0^p f_0, T_0^q g_0)$

$$< \max \begin{Bmatrix} S(f_0, f_0, g_0), \\ S(T_0^p f_0, T_0^p f_0, f_0), \\ S(T_0^q g_0, T_0^q g_0, g_0), \\ S(T_0^q g_0, T_0^q g_0, f_0), \\ S(T_0^p f_0, T_0^p f_0, g_0) \end{Bmatrix}$$

For any given $x_0 \in Y_0$, $\exists$ a positive integer $p(x_0)$, such that,

(3.4) $S\left(T_0{}^{p(x_0)} f_0, T_0{}^{p(x_0)} f_0, T_0^{p(x_0)} g_0\right)$

$$< max \left\{ \begin{matrix} S(f_0, f_0, g_0), S\left(T_0^{p(x_0)} f_0, T_0^{p(x_0)} f_0, f_0\right), \\ S\left(T_0^{\ \ p(x_0)} f_0, T_0^{p(x_0)} f_0, g_0\right), \\ S(T_0^{\ \ p(x_0)} g_0, T_0^{\ \ (x_0)} g_0, f_0), \\ S\left(T_0^{\ \ p(x_0)} f_0, T_0^{p(x_0)} f_0, g_0\right) \end{matrix} \right\}$$

For any given $x_0, y_0 \in Y_0, x_0 \neq y_0$, $\exists$ a positive integer $p(x_0, y_0)$, such that (3.5)

$$S\left(T_0^{p(x_0,y_0)} f_0, T_0^{p(x_0,y_0)} f_0, T_0^{p(x_0,y_0)} g_0\right)$$

$$< max \left\{ \begin{matrix} S(f_0, f_0, g_0), \\ S\left(T_0^{p(x_0,y_0)} f_0, T_0^{p(x_0,y_0)} f_0, f_0\right), \\ S\left(T_0^{\ \ p(x_0,y_0)} f_0, T_0^{p(x_0,y_0)} f_0, g_0\right), \\ S\left(T_0^{p(x_0,y_0)} g_0, T_0^{p(x_0,y_0)} g_0, f_0\right), \\ S\left(T_0^{\ \ p(x_0,y_0)} f_0, T_0^{p(x_0,y_0)} f_0, g_0\right) \end{matrix} \right\}$$

Now we introduce contractive conditions on B_4 - metric spaces.

Definition 3.2. Let (Y_0, B_4) be a B_4-metric space. Suppose T_0 is a self-map on Y_0. We introduce various contractive conditions as follows: for each $f_0, g_0 \in Y_0, f_0 \neq g_0$.

(3.6) $$B_4(T_0 f_0, T_0 f_0, T_0 f_0, T_0 g_0) < max \left\{ \begin{matrix} B_4(f_0, f_0, f_0, g_0), B_4(T_0 f_0, T_0 f_0, T_0 f_0, f_0) \\ B_4(T_0 g_0, T_0 g_0, T_0 g_0, g_0), \\ B_4(T_0 g_0, T_0 g_0, T_0 g_0, f_0), \\ B_4(T_0 f_0, T_0 f_0, T_0 f_0, g_0) \end{matrix} \right\}$$

$\exists$ a positive integer p, such that,

(3.7)

$$B_4(T_0^{\ \ p} f_0, T_0^{\ \ p} f_0, T_0^{\ \ p} f_0, T_0^p g_0) <$$

$$max \left\{ \begin{matrix} B_4(f_0, f_0, f_0, g_0), B_4(T_0^{\ \ p} f_0, T_0^{\ \ p} f_0, T_0^p f_0, f_0), \\ B_4(T_0^{\ \ p} g_0, T_0^{\ \ p} g_0, T_0^{\ \ p} g_0, g_0), \\ B_4(T_0^{\ \ p} g_0, T_0^{\ \ p} g_0, T_0^{\ \ p} g_0, f_0), \\ B_4(T_0^{\ \ p} f_0, T_0 f_0, T_0^{\ \ p} f_0, g_0) \end{matrix} \right\}$$

There exist positive integers p, q such that, (3.8) $B_4(T_0^{\ \ p} f_0, T_0^{\ \ p} f_0, T_0^{\ \ p} f_0, T_0^q g_0) <$

$$max \left\{ \begin{matrix} B_4(f_0, f_0, f_0, g_0), B_4(T_0^{\ \ p} f_0, T_0^{\ \ p} f_0, T_0^p f_0, f_0), \\ B_4(T_0^{\ \ q} g_0, T_0^{\ \ q} g_0, T_0^{\ \ q} g_0, g_0), \\ B_4(T_0^q g_0, T_0^q g_0, T_0^q g_0, f_0), \\ B_4(T_0^{\ \ p} f_0, T_0^{\ \ p} f_0, T_0^{\ \ p} f_0, g_0) \end{matrix} \right\}$$

For any given $x_0 \in Y_0$, $\exists$ a positive integer $p(x_0)$, such that, (3.9)

$$B_4\left(T_0^{p(x_0)}f_0, T_0^{p(x_0)}f_0, T_0^{p(x_0)}f_0, T_0^{p(x_0)}g_0\right) < max\left\{\begin{matrix} B_4(f_0, f_0, f_0, g_0), \\ B_4\left(T_0^{p(x_0)}f_0, T_0^{p(x_0)}f_0, T_0^{p(x_0)}f_0, f_0\right) \\ B_4\left(T_0\ {}^{p(x_0)}g_0, T_0^{p(x_0)}, T_0^{p(x_0)}g_0, g_0\right), \\ B_4\left(T_0^{p(x_0)}g_0, T_0^{p(x_0)}g_0, T_0^{p(x_0)}g_0, f_0\right), \\ B_4\left(T_0^{p(x_0)}f_0, T_0^{p(x_0)}f_0, T_0^{p(x_0)}f_0, g_0\right) \end{matrix}\right\}$$

For any given $f_0, g_0 \in Y_0, f_0 \neq g_0$, $\exists$ a positive integer $p(x_0, y_0) \ni$

(3.10)

$$B_4\left(T_0^{p(x_0,y_0)}f_0, T_0^{p(x_0,y_0)}f_0, T_0^{p(x_0,y_0)}f_0, T_0^{p(x_0,y_0)}g_0\right) <$$

$$\max\left\{\begin{matrix} B_4(f_0, f_0, f_0, g_0), \\ B_4\left(T_0^{p(x_0,y_0)}f_0, T_0^{p(x_0,y_0)}f_0, T_0^{p(x_0,y_0)}f_0, f_0\right) \\ B_4\left(T_0^{p(x_0,y_0)}g_0, T_0^{p(x_0,y_0)}g_0, T_0^{p(x_0,y_0)}g_0, g_0\right) \\ B_4\left(T_0^{p(x_0,y_0)}g_0, T_0^{p(x_0,y_0)}g_0, T_0^{p(x_0,y_0)}g_0, f_0\right) \\ B_4\left(T_0^{p(x_0,y_0)}f_0, T_0^{p(x_0,y_0)}f_0, T_0^{p(x_0,y_0)}f_0, g_0\right) \end{matrix}\right\}$$

Preposition 3.1. Let (Y_0, S) be a S - metric space, $\left(Y_0, B_{4_S}\right)$ be the induced B_4-metric space. Suppose T_0 is a self-map on Y_0. If T_0 satisfies the inequality (3.1), for each $f_0, g_0 \in Y_0, f_0 \neq g_0$. Then T_0 satisfies the inequality (3.6), for each $f_0, g_0 \in Y_0, f_0 \neq g_0$ with B_{4S}

Proof. Let the inequality

$$S(T_0f_0, T_0f_0, T_0g_0) < \ max\left\{\begin{matrix} S(f_0, f_0, g_0), S(T_0f_0, T_0f_0, f_0), \\ S(T_0g_0T_0g_0, g_0), \\ S(T_0g_0, T_0g_0, f_0), \\ S(T_0f_0, T_0f_0, g_0) \end{matrix}\right\}$$

be satisfied,

for each $f_0, g_0 \in Y_0, f_0 \neq g_0$. Then by Lemma 2.2, We have$B_{4_s}(T_0f_0, T_0f_0, T_0f_0, T_0g_0)$

$$= S(T_0f_0, T_0f_0, T_0g_0) + S(T_0f_0, T_0f_0, T_0g_0)$$
$$+S(T_0f_0, T_0f_0, T_0g_0) + S(T_0f_0, T_0f_0, T_0g_0)$$

$= 4S(T_0f_0, T_0f_0, T_0g_0)$

$$< 4\max\left\{\begin{matrix} S(f_0, f_0, g_0), S(f_0, f_0, T_0f_0), \\ S(g_0, g_0, T_0g_0), S(f_0, f_0, T_0g_0), \\ S(g_0, g_0, T_0f_0) \end{matrix}\right\}$$

$$= \max\left\{\begin{matrix} 4S(f_0, f_0, g_0), 4S(f_0, f_0, T_0f_0), \\ 4S(g_0, g_0, T_0g_0), 4S(f_0, f_0, T_0g_0), \\ 4S(g_0, g_0, T_0f_0) \end{matrix}\right\}$$

$$= \max\left\{\begin{matrix} B_{4_s}(f_0,f_0,f_0,g_0), B_{4_s}(f_0,f_0,f_0,T_0f_0), \\ B_{4_s}(g_0,g_0,g_0,T_0g_0), B_{4_s}(f_0,f_0,f_0,T_0g_0), \\ B_{4_s}(g_0,g_0,g_0,T_0g_0) \end{matrix}\right\}$$

$$= \max\left\{\begin{matrix} B_{4_s}(f_0,f_0,f_0,g_0), \\ B_{4_s}(T_0f_0,T_0f_0,T_0f_0,f_0), \\ B_{4_s}(T_0g_0,T_0g_0,T_0g_0,g_0), \\ B_{4_s}(T_0g_0,T_0g_0,T_0g_0,f_0), \\ B_{4_s}(T_0f_0,T_0f_0,T_0f_0,g_0) \end{matrix}\right\}$$

Thus (3.6) is satisfied in $\left(X, B_{4_s}\right)$.

Preposition 3.2. Let (Y_0, B_4) be a B_4 - metric space, (Y_0, S_b) be the S - metric space with index 2 induced by B_4. Suppose T_0 is a self-map on Y_0. If T_0 satisfies the inequality (3.6), then T_0 satisfies the inequality (3.1). for each $f_0, g_0 \in Y_0, f_0 \neq g_0$

Proof. Let the inequality (3.6) be satisfied. Using the inequality (3.6) and, we have

$$S_b(T_0f_0,T_0f_0,T_0g_0) = B_4(T_0f_0,T_0f_0,T_0f_0,T_0g_0) + B_4(T_0f_0,T_0f_0,T_0f_0,T_0g_0)$$
$$+B_4(T_0f_0,T_0f_0,T_0f_0,T_0g_0)$$
$$= 3B_4(T_0f_0,T_0f_0,T_0f_0,T_0g_0)$$
$$< 3max\left\{\begin{matrix} B_4(f_0,f_0,f_0,g_0), B_4(T_0f_0,T_0f_0,T_0f_0,f_0), \\ B_4(T_0g_0,T_0g_0,T_0g_0,g_0), B_4(T_0g_0,T_0g_0,T_0g_0,f_0), \} \\ B_4(T_0f_0,T_0f_0,T_0f_0,g_0) \end{matrix}\right\}$$
$$< max\left\{\begin{matrix} 3B_4(f_0,f_0,f_0,g_0), 3B_4(T_0f_0,T_0f_0,T_0f_0,f_0), \\ 3B_4(T_0g_0,T_0g_0,T_0g_0,g_0), 3B_4(T_0g_0,T_0g_0,T_0g_0,f_0), \\ 3B_4(T_0f_0,T_0f_0,T_0f_0,g_0) \end{matrix}\right\}$$
$$= \max\left\{\begin{matrix} B_4(f_0,f_0,f_0,g_0) + B_4(g_0,g_0,g_0,f_0), \\ B_4(T_0f_0,T_0f_0,T_0f_0,f_0) + B_4(f_0,f_0,f_0,Tf_0), \\ B_4(T_0g_0,T_0g_0,T_0g_0,g_0) + B_4(g_0,g_0,g_0,T_0g_0) \\ B_4(T_0g_0,T_0g_0,T_0g_0,f_0) + B_4(f_0,f_0,f_0,Tg_0) \\ B_4(T_0f_0,T_0f_0,T_0f_0,g_0) + B_4(g_0,g_0,g_0,T_0f_0) \end{matrix}\right\}$$

$$= \max\left\{\begin{matrix} S_bf_0,f_0,g_0), S_b(f_0,f_0,Tf_0), S_b\left(g_{0,},g_0,T_0g_0\right) \\ S_b(f_0,f_0,Tg_0), S_b(g_0,g_0,T_0f_0) \end{matrix}\right\}$$

And so, the inequality (3.1) is satisfied on (Y_0, S_b).

Corollary 3.1. Let (Y_0, S) be a S - metric space, $\left(Y_0, B_{4_s}\right)$ be the B_4– metric space induced by S. Suppose T_0 is a self-map on Y_0. If T_0 satisfies the inequality (3.2). [resp. (3.3), (3.4), and (3.5)], then T_0 satisfies the inequality (3.7) [resp. (3.8), (3.9), and (3.10)].

Corollary 3.2. Let (Y_0, B_4) be a B_4-metric space, (Y_0, S_b) be the metric space obtained by the metric induced by B_4. Suppose T_0 is a self-map on Y_0. If T_0 satisfies the inequality (3.7) [resp. (3.8), (3.9), and (3.10)]. Then T_0 satisfies the inequality (3.2) with S_b [resp. (3.3), (3.4), and (3.5)]

The implications of the following prop can be easily established.

Preposition 3.3. Let (Y_0, B_4) be a B_4 - metric space. Suppose T_0 is a self-mapping of Y_0. Then (3.6) ⇒ (3.7) ⇒ (3.8) and (3.7) ⇒ (3.9) ⇒ (3.10)

The following example shows that (3.10) need not imply (3.9)

Example 3.1. Let $Y_0 = [0,1] \cup \{3\}$ and define the function $B_4: Y_0 \to [0, \infty)$.

Define $B_4(f_0, g_0, h_0, t_0) = |f_0 - h_0| + |g_0 - h_0| + |f_0 + g_0 + h_0 - 3t_0|, \forall f_0, g_0, h_0, t_0 \in Y_0$.

Then, by Example 2.2, B_4 is a B_4 - metric on Y_0. Define $T_0: Y_0 \to Y_0$ by

$$T_0 f_0 = \begin{cases} \sqrt{f_0}, \text{ if } x \in [0,1], & f_0 \neq \frac{1}{2}, f_0 \neq \frac{1}{3} \\ \frac{1}{3}, & \text{if } f_0 = \frac{1}{2} \\ 3, & \text{if } f_0 = \frac{1}{3} \\ \frac{1}{2}, & \text{if } f_0 = 3 \end{cases}.$$

Then T_0 is a self-map on the B_4 - metric space. $Y_0 = [0,1] \cup \{3\}$.

Further, T_0 satisfies (3.10), but does not satisfy (3.9).

4. Fixed point theorems

In this section, we show that under certain contractive conditions a fixed point, if exists, is unique. And we obtain some fixed point theorems for self-maps on a B_4-metric space.

Theorem 4.1. Let (X, B_4) be a B_4 - metric space. Suppose T_0 is a self-map on Y_0 which satisfies the inequality (3.10). If T_0 has at most one fixed point.

Proof. Suppose that f_0 and g_0 are two fixed points of T_0, such that $f_0, g_0 \in Y_0, f_0 \neq g_0$.

Then by inequality (3.10), ∃ a positive integer $p = p(x_0, y_0)$, such that

$$B_4\left(T_0^{p(x_0,y_0)}f_0, T_0^{p(x_0,y_0)}f_0, T_0^{p(x_0,y_0)}f_0, T_0^{p(x_0,y_0)}g_0\right) <$$
$$\max\left\{\begin{array}{c} B_4(f_0,f_0,f_0,g_0), B_4\left(T_0^{p(x_0,y_0)}f_0, T_0^{p(x_0,y_0)}f_0, T_0^{p(x_0,y_0)}f_0, f_0\right) \\ B_4\left(T_0^{p(x_0,y_0)}g_0, T_0^{p(x_0,y_0)}g_0, T_0^{p(x_0,y_0)}g_0, g_0\right), \\ B_4\left(T_0^{p(x_0,y_0)}g_0, T_0^{p(x_0,y_0)}g_0, T_0^{p(x_0,y_0)}g_0, f_0\right), \\ B_4\left(T_0^{p(x_0,y_0)}f_0, T_0^{p(x_0,y_0)}f_0, T_0^{p(x_0,y_0)}f_0, g_0\right) \end{array}\right\}$$
$$= \max\{B_4(f_0,f_0,f_0,g_0), 0, 0, B_4(f_0,f_0,f_0,g_0), B_4(g_0,g_0,g_0,f_0)\}$$
$= B_4(f_0,f_0,f_0,g_0)$ (by lemma 2.1)

Therefore $B_4(f_0,f_0,f_0,g_0) < B_4(f_0,f_0,f_0,g_0)$, a contradiction.

Hence $f_0 = g_0$. Therefore, T_0 has at most one fixed point.

In view of Proposition 3.3, we have the following corollary.

Corollary 4.1. Let (Y_0, B_4) be an B_4 - metric space. Suppose T_0 is a self-map on Y_0, satisfying the inequality (3.6). [resp. (3.7)] has at most one fixed point.

Proof. T_0 satisfies (3.6) ⇒ T_0 satisfies (3.10)

⇒ a fixed point, if exists, is unique.

Theorem 4.2. Let (Y_0, B_4) be a B_4 - metric space. Suppose T_0 is a self-map on Y_0, satisfying the inequality (3.8). If T_0 has at most one fixed point.

Proof. Suppose that f_0 and g_0 are two fixed points of T_0.

Then by inequality (3.8), there exist positive integers p, q such that,

$$B_4(f_0,f_0,f_0,g_0) = B_4\left(T_0^p f_0, T_0^p f_0, T_0^p f_0, T_0^q g_0\right)$$
$$< max\left\{\begin{array}{c} B_4(f_0,f_0,f_0,g_0), B_4\left(T_0^p f_0, T_0^p f_0, T_0^p f_0, f_0\right), \\ B_4\left(T_0^q g_0, T_0^q g_0, T_0^q g_0, g_0\right), B_4\left(T_0^q g_0, T_0^q g_0, T_0^q g_0, f_0\right), \\ B_4\left(T_0^p f_0, T_0^p f_0, T_0^p f_0, g_0\right) \end{array}\right\}$$
$$= \max\{B_4(f_0,f_0,f_0,g_0), 0, 0, B_4(g_0,g_0,g_0,f_0), B_4(f_0,f_0,f_0,g_0)\}$$
$= B_4(f_0,f_0,f_0,g_0)$ (by lemma 2.1)

Therefore, $B_4(f_0,f_0,f_0,g_0) < B_4(f_0,f_0,f_0,g_0)$,

which is a contradiction. Hence $f_0 = g_0$.

Definition 4.1. Let (Y_0, B_4) be a B_4-metric space and $A \subset Y_0$. Then, A is called B_4 - bounded if $\exists r > 0$, such that $B_4(f_0,f_0,f_0,g_0) < r, \forall f_0, g_0 \in A$.

Definition 4.2. Let (Y_0, B_4) be a B_4-metric space. Suppose T_0 is a self-mapping of Y_0, and $f_0 \in Y_0$. A point f_0 is called a periodic point of T_0, if $\exists$ a positive integer n, such that $T_0{}^n f_0 = f_0$. The least positive integer satisfying the condition $T_0{}^n f_0 = f_0$ is called the periodic index of f_0.

Lemma 4.1. Let (Y_0, B_4) be a B_4-metric space. Suppose $\{f_{0_n}\}$ and $\{g_{0_n}\}$ or such that $\lim_{n\to\infty} f_{0_n} = f_0$ and $\lim_{n\to\infty} g_{0_n} = g_0$. Then $B_4(f_{0_n}, f_{0_n}, f_{0_n}, g_{0_n}) \to B_4(f_0, f_0, f_0, g_0)$

Lemma 4.2. Let (Y_0, B_4) be a B_4-metric space. Suppose $\{f_{0_n}\}$ converges to f_0 and $g_0 \in Y_0$. Then, $B_4(f_{0_n}, f_{0_n}, f_{0^n}, g_{0_n}) \to B_4(f_0, f_0, f_0, g_0)$.

Proof. Take $g_{0_n} = g_0, \forall n$, in the Lemma 2.4.

Definition 4.3. Let (Y_0, B_4) be a B_4-metric space. Suppose T_0 is self-map on Y_0, and $A \subset Y_0, f_0 \in Y_0$.

Then (1). $\delta(A) = \sup\{B_4(f_0, f_0, f_0, g_0): f_0, g_0 \in A\}$.

(2) $\mathcal{O}_{T_0}(f_0, n) = \{f_0, T_0 f_0, T_0^2 f_0, \dots, T_0{}^n f_0\}$.

(3) $\mathcal{O}_{T_0}(f_0, \infty) = \{f_0, T_0 f_0, T_0^2 f_0, \dots, T_0{}^n f_0, \dots\}$.

$\delta(A)$ is called diameter of A.

If $\delta(A) < \infty$, then we say that A is bounded.

Theorem 4.3. Let (Y_0, B_4) be a B_4-metric space. Suppose T_0 is a self-mapping of Y_0, such that Every Cauchy sequence of the form $\{T_0{}^n f_0\}$ is convergent in $Y_0, \forall f_0 \in Y_0$; $\exists h \in [0,1)$ such that

$$B_4(T_0 f_0, T_0 f_0, T_0 f_0, T_0 g_0) \le \max \left\{ \begin{array}{c} B_4(f_0, f_0, f_0, g_0), B_4(T_0 f_0, T_0 f_0, T_0 f_0, f_0), \\ B_4(T_0 g_0, T_0 g_0, T_0 g_0, g_0), B_4(T_0 g_0, T_0 g_0, T_0 g_0, f_0), \\ B_4(T_0 f_0, T_0 f_0, T_0 f_0, g_0) \end{array} \right\}$$

for each $f_0, g_0 \in Y_0$. Then $B_4(T_0{}^i f_0, T_0{}^i f_0, T_0{}^i f_0, T_0{}^j f_0) \le h\delta[\mathcal{O}_{T_0}(f_0, n)]$ $\forall i, j \le n, n \in \mathbb{N}$ and $f_0 \in Y_0$;

$\delta[\mathcal{O}_{f_0}(f_0, \infty)] \le \frac{3}{1-h} B_4(T_0 f_0, T_0 f_0, T_0 f_0, f_0) \forall f_0 \in Y_0$;

$f_0 \in Y_0 \Rightarrow \{T_0{}^n f_0\}$ is a Cauchy sequence;

$\lim_{n\to\infty} T_0^n f_0 = g_0$, T_0 has one and only one fixed point.

Proof. (i) For each $f_0 \in Y_0$ and all $1 \le i, j \le n, n \in \mathbb{N}$

We have $T_0{}^{i-1}, T_0{}^{i}, T_0{}^{j-1}, T_0{}^{j} \in \mathcal{O}(f_0, n)$

It follows from (3.1)$B_4(T_0{}^i f_0, T_0{}^i f_0, T_0{}^i f_0, T_0^j f_0)$

$$= B_4\left(T_0(T_0^{i-1}f_0), T_0(T_0^{i-1}f_0), T_0(T_0^{i-1}f_0), T_0(T_0^{j-1}f_0)\right)$$

$$\le h\max\begin{Bmatrix} B_4(T_0{}^{i-1}f_0, T_0{}^{i-1}f_0, T_0{}^{i-1}f_0, T_0{}^{j-1}f_0) \\ B_4(T_0{}^{i-1}f_0, T_0{}^{i-1}f_0, T_0{}^{i-1}f_0, T_0{}^{i}f_0) \\ B_4(T_0{}^{j-1}f_0, T_0{}^{j-1}f_0, T_0{}^{j-1}f_0, T_0{}^{j}f_0) \\ B_4(T_0{}^{i-1}f_0, T_0{}^{i-1}f_0, T_0^{i-1}f_0, T_0{}^{j}f_0) \\ B_4(T_0{}^{j-1}f_0, T_0^{j-1}f_0, T_0{}^{j-1}f_0, T_0{}^{i}f_0) \end{Bmatrix}$$

$$\le h\delta[\mathcal{O}_{T_0}(f_0, n)]$$

$$B_4(T_0{}^i f_0, T_0{}^i f_0, T_0{}^i f_0, T_0{}^j f_0) =\le h\delta[\mathcal{O}_{T_0}(f_0, n)]$$

This shows that $\exists j, 1 \le j \le n$ such that $\delta[\mathcal{O}_{T_0}(f_0, n)] = B_4(f_0, f_0, f_0, T_0^j f_0)$

(ii) We have $\mathcal{O}_{T_0}(f_0, \infty) = \{f_0, T_0^2 f_0, \dots, T_0{}^n f_0, \dots\}$ $\delta[\mathcal{O}_{T_0}(f_0, n)] = B_4(f_0, f_0, f_0, T_0{}^j f_0)$ for some $1 \le k \le n$

Also $\delta[\mathcal{O}_{T_0}(f_0, n)] \le \delta[\mathcal{O}_{T_0}(f_0, n+1)] = B_4(f_0, f_0, f_0, T_0^l f_0)$ for some $1 \le l \le n+1$. Given f_0 and n, $\exists k, 1 \le k \le n$ such that

$\mathcal{O}_{T_0}(f_0, \infty) = \sup[\mathcal{O}_{T_0}(f_0, n)] = \frac{\sup}{k \to \infty} B_4(f_0, f_0, f_0, f_0, f_0, f_{0_k}) n < m = n + k$ (say)

$$B_4(f_{0_n}, f_{0_n}, f_{0_n}, f_{0_{n+k}}) \le h\delta[\mathcal{O}_{T_0}(f_{0_{n-1}}, n + k_1)], n + k_1 \le m$$

Therefore $\delta[\mathcal{O}_{T_0}(f_{0_n}, m)] = B_4\left(f_{0_n}, f_{0_n}, f_{0_n}, f_{0_{n+k_1}}\right)$

$\le h\delta[\mathcal{O}_{T_0}(f_{0_{n-1}}, n + k_1)]$

$$\le h^2\delta[\mathcal{O}_{T_0}(f_{0_{n-2}}, n + k_2)], n + k_2 \le n + k_1$$

Now $B_4(f_0, f_0, f_0, T_0{}^k f_0) \le B_4(f_0, f_0, f_0, T_0 f_0) + B_4(f_0, f_0, f_0, T_0 f_0) + B_4(f_0, f_0, f_0, T_0 f_0) + B_4(T_0{}^k f_0, T_0{}^k f_0, T_0{}^k f_0, f_0)$

$$\delta[\mathcal{O}_{T_0}(f_0, n)] \le 3B_4(f_0, f_0, f_0, T_0 f_0) + B_4(T_0{}^k f_0, T_0{}^k f_0, T_0{}^k f_0, f_0)$$

$$\le 3B_4(f_0, f_0, f_0, T_0 f_0) + h\mathcal{O}_{T_0}(f_0, k)$$

$$\le 3B_4(f_0, f_0, f_0, T_0 f_0) + h\mathcal{O}_{T_0}(f_0, n)$$

$$\delta[\mathcal{O}_{T_0}(f_0, n)] - h\mathcal{O}_{T_0}(f_0, n) \le 3B_4(f_0, f_0, f_0, T_0 f_0)$$

$$\delta[\mathcal{O}_{T_0}(f_0, n)] \le \frac{3}{1-h} B_4(f_0, f_0, f_0, T_0 f_0), \forall n$$

Therefore $\delta[\mathcal{O}_{T_0}(f_0, n)] \le \frac{3}{1-h} B_4(f_0, f_0, f_0, T_0 f_0), \forall f_0 \in Y_0$.

(iii) $\mathcal{O}_{T_0}(f_0, n) = \{f_0, T_0^2 f_0, \dots, T_0{}^n f_0\}$

Given x and m, $\exists k, 1 \leq k \leq n$, such that $\delta[O_{T_0}(f_0, n)] = B_4(f_0, f_0, f_0, f_{0_k}), n < m$

$$B_4(f_{0_n}, f_{0_n}, f_{0_n}, f_{0_{n+1}}) \leq h\delta[O_{T_0}(f_{0_{n-1}}, n + k_1)], n + 1 \leq k$$

$$\delta[O_{T_0}(f_{0_n}, m)] = B_4\left(f_{0_n}, f_{0_n}, f_{0_n}, f_{0_{n+k_1}}\right) \leq h\delta[O_{T_0}(f_{0_{n-1}}, n + k_1)], n + k_1 \leq m$$

$\leq h^2\delta[O_{T_0}(f_{0_{n-2}}, n + k_2)], n + k_2 \leq n + k_1$

$$\leq h^n\delta[O_{T_0}(f_{0_{n-n}}, f_0 + k_m)], n + k_n \leq n + k_{n-1} \leq m$$

$\leq h^n\delta[O_{T_0}(f_0, n)]$

$\leq h^n\delta[O_{T_0}(f_0, \infty)]$

$\leq h^n \frac{3}{1-h} B_4(T_0f_0, T_0f_0, T_0f_0, f_0)$

$\rightarrow 0$ as $n \rightarrow \infty$

Therefore, $\delta[O_{T_0}(f_{0_n}, m)] \rightarrow 0$ as $m \rightarrow \infty$

Therefore, $B_4(f_{0_n}, f_{0_n}, f_{0_n}, f_{0_m}) \leq \delta[O_{T_0}(f_{0_n}, m)] \rightarrow 0$ as $m \rightarrow \infty$

Therefore, $\{f_{0_n}\}$ is a Cauchy sequence.

(iv) Suppose $\{T_0 \quad ^n f_0\}$ is a Cauchy sequence and hence convergent, say to g_0(by (i)).

That is $\lim_{n\rightarrow\infty} T_0 \quad ^n f_0 = g_0$Write $f_{0_n} = T_0^n f_0$

$T_0(f_{0_n}) = T_0(T_0 \quad ^n f_{0_n}) = T_0(T_0 \quad ^{n+1} f_{0_n}) = f_{0_{n+1}}$

$B_4(T_0(T_0 \quad ^n f_0), T_0(T_0 \quad ^n f_0), T_0(T_0 \quad ^n f_0), T_0 y)$.

$$\leq h\max\left\{\begin{array}{c} B_4(T_0 \quad ^n f_0, T_0 \quad ^n f_0, T_0 \quad ^n f_0, g_0), \\ B_4(T_0(T_0 \quad ^n f_0), T_0(T_0 \quad ^n f_0), T_0(T_0 \quad ^n f_0), T_0 \quad ^n f_0), \\ B_4(T_0g_0, T_0g_0, T_0g_0, g_0), B_4(T_0g_0, T_0g_0, T_0g_0, T_0 \quad ^n f_0), \\ B_4(T_0(T_0 \quad ^n f_0), T_0(T_0 \quad ^n f_0), T_0(T_0 \quad ^n f_0), g_0) \end{array}\right\}$$

$$= h\max\left\{\begin{array}{c} B_4(f_{0_n}, f_{0_n}, f_{0_n}, g_0), B_4(f_{0_{n+1}}, f_{0_{n+1}}, f_{0_{n+1}}, f_{0_m}), \\ B_4(T_0g_0, T_0g_0, T_0g_0, g_0), \\ B_4(T_0g_0, T_0g_0, T_0g_0, f_{0_n}), B_4(f_{0_{n+1}}, f_{0_{n+1}}, f_{0_{n+1}}, g_0) \end{array}\right\}$$

$$= h\max\left\{\begin{array}{c} B_4(f_{0_n}, f_{0_n}, f_{0_n}, y), B_4(f_{0_{n+1}}, f_{0_{n+1}}, f_{0_{n+1}}, f_{0_n}), \\ B_4(g_0, g_{0,}, g_{0,}, T_0g_0), \\ B_4(f_{0_n}, f_{0_n}, f_{0_n}, T_0g_0), B_4(x_{n+1}, x_{n+1}, x_{n+1}, g_0) \end{array}\right\}$$

$= h\max\{0,0, B_4(g_0, g_0, g_0, T_0g_0), B_4(g_0, g_0, g_0, T_0g_0), 0\}$

$= hB_4(g_0, g_0, g_0, T_0g_0)$

L.H.S $B_4(T_0(T_0 \quad ^n f_0), T_0(T_0 \quad ^n f_0), T_0(T_0 \quad ^n f_0), T_0 \quad ^n f_0) = B_4(g_0, g_0, g_0, T_0g_0)$

$\leq hB_4(g_0, g_0, g_0, T_0g_0)$

Therefore, $(1 - h)B_4(g_0, g_0, g_0, T_0g_0) = 0$

Therefore, $B_4(g_0, g_0, g_0, T_0g_0) = 0$, since $0 \leq h < 1$

Therefore $T_0g_0 = g_0$.

Now we prove that g_0 is the one and only one fixed point of T_0. Let g_0 and $g_0{}'$ be two fixed points of T_0.

Then $T_0 g_0 = g_0$ and $T_0 g_0{}' = g_0{}'$. Using (3.1) and Lemma 2.1, we have

$$B_4(g_0, g_0, g_0, g_0') = B_4(T_0 g_0, T_0 g_0, T_0 g_0, T_0 g_0')$$
$$\leq h\max\left\{\begin{array}{c} B_4(g_0, g_0, g_0, g_0'), B_4(g_0, g_0, g_0, T_0 g_0), \\ B_4(g_0', g_0', g_0', T_0 g_0{}'), \\ B_4(g_0, g_0, g_0, T_0 g_0'), B_4(g_0', g_0{}', g_0, T_0 g_0) \end{array}\right\}$$
$$= hB_4(g_0, g_0, g_0, g_0)$$

Since $0 \leq h < 1$, we get $B_4(g_0, g_0, g_0, g_0') = 0$, that is, $T_0 g_0 = T_0 g_0{}' \Rightarrow g_0 = g_0{}'$

Therefore T_0 has one and only one fixed point g_0.

Also $\lim_{n\to\infty} T^n f_0 = g_0$.

Theorem 4.4. Let (Y_0, B_4) be an B_4-metric space. Suppose T_0 is a self-map on Y_0, which satisfies the inequality (3.10), and $f_0 \in Y_0$. Assume that x is a periodic point of T_0 with periodic index $m \geq 2$.

Suppose

$$B_4\left(T_0^{p(x_0,y_0)} f_0, T_0^{p(x_0,y_0)} f_0, T_0^{p(x_0,y_0)} f_0, T_0^{p(x_0,y_0)} g_0\right)$$
$$< max\left\{\begin{array}{c} B_4(f_0, f_0, f_0, g_0), B_4\left(T_0^{p(x_0,y_0)} f_0, T_0^{p(x_0,y_0)} f_0, T_0^{p(x_0,y_0)} f_0, f_0\right) \\ B_4\left(T_0^{p(x_0,y_0)} g_0, T_0^{p(x_0,y_0)} g_0, T_0^{p(x_0,y_0)} g_0, g_0\right), \\ B_4\left(T_0^{p(x_0,y_0)} g_0, T_0^{p(x_0,y_0)} g_0, T_0^{p(x_0,y_0)} g_0, f_0\right), \\ B_4\left(T_0^{p(x_0,y_0)} f_0, T_0^{p(x_0,y_0)} f_0, T_0^{p(x_0,y_0)} f_0, g_0\right) \end{array}\right\}$$

for each $f_0, g_0 \in Y_0, f_0 \neq g_0$. for any $T_0{}^i f_0, T_0{}^j f_0 \in \{T_0{}^n f_0\}$

$(n \geq 0)$, $T_0{}^i f_0 \neq T_0{}^j f_0 \exists T_0{}^k f_0, T_0^l f_0 \in \{T_0{}^n f_0\}$, such that

$T_0{}^{(T_0{}^k f_0, T_0{}^l f_0)}(T_0{}^k f_0) = T_0{}^i f_0$ and $T_0^{p(T_0{}^k f_0, T_0{}^l f_0)}\left(T_0^l f_0\right) = T_0^j f_0$.

Then, $m = 1$, thus the point f_0 is a fixed point of T_0 in Y_0.

Proof. The given inequality (3.10) is Suppose f_0 is a periodic point of T_0 with periodic index $m \geq 2$,

show that $T_o f_0 = f_0$. $\{T_0{}^n f_0\} = \{f_0, T_0 f_0, \dots, T_0{}^{m-1} f_0\} = A$ (say), Since $f_0 \neq T_0 f_0$,

then there exist $T_0{}^i f_0, T_0{}^j f_0 \in A, \Rightarrow T_0{}^i f_0 \neq T_0^j f_0$, such that

$$\delta(A) = B_4\left(T_0{}^i f_0, T_0{}^i f_0, T_0{}^i f_0, T_0{}^j f_0\right)$$

By the hypothesis, there exist $T_0{}^k f_0, T_0^l f_0 \in A$, such that $T_0{}^{p(T_0{}^k f_0, T_0{}^l f_0)}(T_0{}^k f_0) = T_0^i f_0$ and $T_0{}^{p(T_0{}^k f_0, T_0{}^l f_0)}(T_0{}^l f_0) = T_0{}^j f_0$ Since $T_0{}^i f_0 \neq T_0{}^j f_0$, we obtain $T_0{}^k f_0 \neq T_0{}^l f_0$.

Hence, we have $\delta(A) = B_4(T_0{}^i f_0, T_0{}^i f_0, T_0{}^i f_0, T_0{}^j f_0)$

$$= B_4\begin{pmatrix} T_0^{p(T_0{}^k f_0, T_0{}^l f_0)}(T_0{}^k f_0), T_0{}^{p(T_0{}^k f_0, T_0{}^l f_0)}(T_0{}^k f_0), \\ T_0{}^{p(T_0{}^k f_0, T_0{}^l f_0)}(T_0{}^k f_0), T_0{}^{p}(T_0{}^k f_0, T_0{}^l f_0)(T_0{}^l f_0) \end{pmatrix}$$

$$< max\begin{Bmatrix} B_4(T_0{}^k f_0, T_0{}^k f_0, T_0^k f_0, T_0{}^l f_0), B_4(T_0{}^i f_0, T_0{}^i f_0, T_0{}^i f_0, T_0{}^k f_0), \\ B_4(T_0^j f_0, T_0{}^j f_0, T_0{}^j f_0, T_0{}^l f_0), B_4(T_0{}^j f_0, T_0^j f_0, T_0^j f_0, T_0{}^k f_0), \\ B_4(T_0{}^i f_0, T_0{}^i f_0, T_0{}^i f_0, T_0{}^l f_0) \end{Bmatrix}$$

$\leq \delta(A)$, a contradiction, and so, we have $f_0 = T_0 f_0$.

Hence $m = 1$, consequently, $T_o f_0 = f_0$ and f_0 is a fixed point of T_0

Corollary 4.2. Let (Y_0, B_4) be a B_4-metric space. Suppose T_0 is a self-mapping of Y_0, the inequality (3.9) be satisfied, and $f_0 \in Y_0$ be a periodic point of T_0, with periodic index m, satisfying (4.1)

Then, $m = 1$ and the point f_0 is only one fixed point of T_0 in Y_0.

Proof. Write $A = \{f_0, T_0 f_0, \dots, T_0{}^{m-1} f_0\}$

Take $p = p(f_0)$ (associated with x_0 according to (3.9)).

since (3.9) $\Longrightarrow$ (3.10), we get the result from Theorem 4.4.

Corollary 4.3. Let (Y_0, B_4) be a B_4-metric space. Suppose T_0 is a self-map on Y_0, which satisfies the inequality (3.8), and $f_0 \in Y_0$ be a periodic point of T_0. Then, f_0 is the at most fixed point of T_0 if $\exists T_0{}^{n_3} f_0, T_0{}^{n_4} f_0 \in \{T_0{}^n f_0\}(n \geq 0)$, and $T_0{}^{n_3} f_0 \neq T_0{}^{n_4} f_0$, such that $T_0{}^p(T_0{}^{n_3} f_0) = T_0{}^{n_1} f_0$ and $T_0^q(T_0{}^{n_4} f_0) = T_0{}^{n_2} f_0$, for any $T_0{}^{n_1} f_0, T_0{}^{n_2} f_0 \in \{T_0{}^n f_0\}(n \geq 0), T_0{}^{n_1} f_0 \neq T_0{}^{n_2} f_0$. Here, p and q are the positive integers.

The proof is similar to that of Theorem 4.4.

Corollary 4.4. Let (Y_0, B_4) be a B_4-metric space. Suppose T_0 is a self-map on Y_0, which satisfies the inequality (3.7). Then, the following conditions are equivalent:

T_0 has a fixed point in $Y_0 \exists$ a periodic point $f_0 \in Y_0$ of T_0 and (4.1) holds for some $f_0 \in Y_0$.

Then, the point f_0 is only one fixed point of T_0 in Y_0.

Proof. Clearly (1) ⇒ (2).

Now suppose (2) holds.

Then, (3.7) ⇒ (3.10) and hence by Theorem 4.4, we get the result.

Theorem 4.5. Let (Y_0, B_4) be a B_4-metric space. Suppose T_0 is a selfmap on Y_0, which satisfies the inequality (3.8). Suppose $f_0 \in Y_0$ is a periodic point of T_0 with periodic index m. Suppose that p and q are positive integers. Suppose the following conditions are satisfied.

(i) $p = p_1 m + p_2, \quad q = q_1 m + q_2, \quad 0 \le p_2, q_2 < m,$

and p_1, q_1 are non-negative integers

(ii) $2|p_2 - q_2| \neq m$ Then, the point f_0 is the at most one fixed point of T_0 in Y_0.

Proof. If $m = 1$ then $T_0 f_0 = T_0^m f_0$ and hence x is a fixed point of T_0. Hence, we assume that $m > 1$, now we show that x is a fixed point of T_0. Let

$A = \{T_0^n f_0\} = \{f_0, T_0 f_0, T_0^2 f_0, \dots, T_0^n f_0, \dots\}$

Since the periodic index of f_0 is m, we have $A = \{T_0{}^n f_0\} = \{f_0, T_0 f_0, T_0^2 f_0, \dots, T_0{}^{m-1} f_0\}$ and the elements in A are distinct. Therefore, there exist i, j, such that $0 \le i < j < m$

$T_0{}^m f_0 = f_0$ and $T_0^i f_0 \neq f_0$ for $2 \le i < m$

$\Longrightarrow T_0(T_0^m f_0) = T_0 f_0$

by the given inequality (3.8)

$$B_4\left(T_0^p f_0, T_0^p f_0, T_0^p f_0, T_0^q g_0\right) < \max \left\{ \begin{array}{c} B_4(f_0, f_0, f_0, g_0), \\ B_4\left(T_0^p f_0, T_0^p f_0, T_0^p f_0, f_0\right), \\ B_4\left(T_0^q g_0, T_0{}^q g_0, T_0^q g_0, g_0\right), \\ B_4\left(T_0^q g_0, T_0{}^q g_0, T_0^q g_0, f_0\right), \\ B_4\left(T_0{}^p f_0, T_0^p f_0, T_0{}^p f_0, g_0\right) \end{array} \right\}$$

Write $i = p_2 + n_1$ where $n_1 = i - p_2$ if $i \ge p_2$ and

$n_1 = m + i - p_2$, if $i < p_2$and $j = q_2 + n_2$ where $n_2 = j - q_2$ if $j \ge q_2$ and $n_2 = m + j - q_2$ if $j < q_2$

Where p_1, p_2, q_2, q_2 are asin(1) Then$0 \le n_1 < m$ and $0 \le n_2 < m$

$$\delta(A) = B_4\left(T_0^{\ i} f_0, T_0^{\ \mathrm{i}} f_0, T_0^{\ \mathrm{i}} f_0, T_0^{\ \mathrm{j}} f_0\right)$$
$$= B_4\left(T_0^{p_2}\left(T_0^{n_1} f_0\right), T_0^{p_2}\left(T_0^{n_1} f_0\right), T_0^{p_2}\left(T_0^{n_1} f_0\right), T_0^{q_2}\left(T_0^{n_2} f_0\right)\right)$$
$$= B_4\left(T_0^{\ p}\left(T_0^{n_1} f_0\right), T_0^{\ p}\left(T_0^{n_1} f_0\right), T_0^{p}\left(T_0^{n_1} f_0\right), T_0^{\mathrm{q}}\left(T_0^{n_2} f_0\right)\right)$$
$$< max\left\{\begin{array}{c} B_4\left(T_0^{n_1} f_0, T_0^{n_1} f_0, T_0^{n_1} f_0, T_0^{\ n_2} f_0\right), \\ B_4\left(T_0^{\ q_2}\left(T_0^{\ n_1} f_0\right), T_0^{\ p}\left(T_0^{\ n_1} f_0\right), T_0^{\ p}\left(T_0^{\ n_1} f_0\right), T_0^{\ n_1} f_0\right), \\ B_4\left(T_0^{q_2}\left(T_0^{\ n_2} f_0\right), T_0^{q_2}\left(T_0^{\ 2} f_0\right), T_0^{q_2}\left(T_0^{\ n_2} f_0\right), T_0^{\ n_2} f_0\right), \\ B_4\left(T_0 q_2\left(T_0^{\ n} f_0\right), T_0 q_2\left(T_0^{n_2} f_0\right), T_0^{\ q_2}\left(T_0^{\ n_2} f_0\right), T_0^{\ n_1} f_0\right), \\ B_4\left(T_0^{\ p}\left(T_0^{\ n_1} f_0\right), T_0^{\ p}\left(T_0^{\ n_1} f_0\right), T_0^{\ p}\left(T_0^{\ n_1} f_0\right), T_0^{\ n_2} f_0\right) \end{array}\right\}$$

$\leq \delta(A)$

Therefore $\delta(A) < \delta(A)$, which is a contradiction. Hence $m > 1$ is false. Consequently $m = 1$ so that $T_0 f_0 = 1$.

Hence f_0 is the one and only one fixed point of T_0 in Y_0.

5. Some applications of Contractive mappings on $B_4 -$ metric spaces

Lemma 5.1. Let (Y_0, B_4) be a complete B_4 - metric space.

Then, $B_4(f_0, f_0, f_0, t_0) - 3B_4(f_0, f_0, f_0, b) \leq B_4(f_0, f_0, f_0, b)$

Proof. From definition 2.1 (ii), $B_4(f_0, g_0, h_0, t_0)$

$$\leq B_4(f_0, f_0, f_0, a_0) + B_4(g_0, g_0, g_0, a_0) + B_4(h_0, h_0, h_0, a_0) + B_4(t_0, t_0, t_0, a_0)$$

Putting $f_0 = g_0 = h_0 = b, t_0 = c$ then,

$B_4(b, b, b, c) \leq B_4(b, b, b, a) + B_4(b, b, b, a) + B_4(b, b, b, a) + B_4(c, c, c, a)$.

$B_4(b, b, b, c) \leq 3B_4(b, b, b, a) + B_4(c, c, c, a)$

$B_4(b, b, b, c) - 3B_4(b, b, b, a) \leq B_4(c, c, c, a)$

Therefore $B_4(f_0, f_0, f_0, t_0) - 3B_4(f_0, f_0, f_0, b) \leq B_4(f_0, f_0, f_0, b)$.

Now, we introduce the notion of a T_b - orbitally complete B_4-metric space.

Definition 5.1. Let (Y_0, B_4) be a B_4 - metric space. Suppose T_0 is a self-mapping of Y_0. Then Y_0 is said to be T_b - orbitally complete if every Cauchy sequence which is contained in the $\{f_0, T_0 f_0, \ldots, T_0^{\ \mathrm{n}} f_0, \ldots\}$, for some $f_0 \in Y_0$ converges in Y_0.

Theorem 5.1. Let (Y_0, B_4) be a B_4 - metric space and $T_0: Y_0 \rightarrow Y_0$. Assume that T_b - orbitally complete. Suppose $\exists\ h \in [0,1) \ni$

(5.1) $B_4(T_0f_0, T_0f_0, T_0f_0, T_0g_0) \leq$

$$h\max\begin{Bmatrix} B_4(f_0,f_0,f_0,g_0), B_4(T_0f_0,T_0f_0,T_0f_0,f_0) \\ B_4(T_0f_0,T_0f_0,T_0f_0,g_0), B_4(T_0g_0,T_0g_0,T_0g_0,f_0), \\ B_4(T_0g_0,T_0g_0,T_0g_0,g_0), \end{Bmatrix} \forall f_0, g_0 \in Y_0.$$

Then, T_0 has only one fixed point in Y_0.

Proof. Since Y_0 is T_b - orbitally continuous condition (1) of Theorem 4.3 is satisfied. Hence the result follows from Theorem 4.3.

The following theorems are the generalizations of the fixed point theorems given in [11] to a B_4 - metric space (Y_0, B_4).

Theorem 5.2. Let (Y_0, B_4) be a complete B_4 - metric space. Suppose T_0 is a continuous self-map on Y_0. There exist positive integers p and q and $0 \leq h < 1$, such that,

(5.2) $B_4\left(T_0^{\ p} f_0, T_0^{\ p} f_0, T_0^p f_0, T_0^{\ q} g_0\right)$

$$\leq h\max\begin{Bmatrix} B_4(T_0^r f_0, T_0^{\ r} f_0, T_0^{\ r} f_0, T_0^{\ s} g_0), B_4\left(T_0^{\ r} f_0, T_0^{\ r} f_0, T_0^{\ r} f_0, T_0^{\ r'} g_0\right) \\ B_4\left(T_0^{\ s} g_0, T_0^{\ s} g_0, T_0^{\ s} g_0, T_0^{\ s'} g_0\right) \\ : 0 \leq r, r' \leq p \text{ and } 0 \leq s, s' \leq q \end{Bmatrix}$$

Then, T_0 has one and only one fixed point in Y_0.

Proof. We may assume that $\frac{1}{4} \leq h < 1$, so that $\frac{h}{1-3h} \geq 1$

Let us assume that $p \geq q$. Suppose $f_0 \in Y_0$,

Assume that sequence $\{T_0^n f_0 : n = 1,2, \dots\}$ is unbounded.

Hence the sequence $\{B_4(T_0^{\ n} f_0, T_0^{\ n} f_0, T_0^{\ n} f_0, T_0^{\ q} f_0) : n = 1,2, \dots\}$ is unbounded.

Hence, $\exists$ an integer n such that

$B_4(T_0^{\ n} f_0, T_0^{\ n} f_0, T_0^{\ n} f_0, T_0^{\ q} f_0) \geq \frac{h}{1-3h} \max\left\{B_4\left(T_0^{\ i} f_0, T_0^{\ i} f_0, T_0^{i} f_0, T_0^{q} f_0\right) : 0 \leq i \leq p\right\}$

Let m be the smallest integer n satisfying (i)

Then, (ii)

$B_4(T_0^{\ m} f_0, T_0^{\ m} f_0, T_0^{\ m} f_0, T_0^{\ q} f_0) > \frac{h}{1-3h} \max\left\{B_4\left(T_0^{\ i} f_0, T_0^{\ i} f_0, T_0^{\ i} f_0, T_0^{q} f_0\right) : 0 \leq i \leq p\right\}$

Clearly, $m > p$. Hence, (iii) $B_4\left(T_0^m f_0, T_0^m f_0, T_0^m f_0, T_0^q f_0\right)$

$$> \frac{h}{1-3h} \max\left\{B_4\left(T_0^{\ i} f_0, T_0^{\ i} f_0, T_0^{i} f_0, T_0^{q} f_0\right) : 0 \leq i \leq p\right\}$$
$$\geq \max\{B_4(T_0^{\ r} f_0, T_0^{\ r} f_0, T_0^{\ r} f_0, T_0^{\ q} f_0) : 0 \leq r < m\}$$

(iii) Consequently, we

have, $(1-3h)B_4(T_0{}^m f_0, T_0{}^m f_0, T_0{}^m f_0, T_0{}^q f_0) >$

$h\max\left\{\begin{matrix} B_4(T_0{}^m f_0, T_0{}^m f_0, T_0{}^m f_0, T_0{}^q f_0) \\ : 0 \le i \le p \end{matrix}\right\}$

$$\ge h\max\left\{\begin{matrix} B_4\left(T_0{}^i f_0, T_0{}^i f_0, T_0{}^i f_0, T_0{}^r f_0\right) - 3B_4(T_0{}^r f_0, T_0^r f_0, T_0{}^r f_0, T_0{}^q f_0) \\ : 0 \le i \le p, 0 \le r < m \end{matrix}\right\}$$

(by Lemma 5.1)

$$\ge h\max\left\{\begin{matrix} B_4\left(T_0{}^i f_0, T_0{}^i f_0, T_0{}^i f_0, T_0{}^r f_0\right) - 3B_4(T_0{}^m f_0, T_0{}^m f_0, T_0{}^m f_0, T_0{}^q f_0) \\ : 0 \le i \le p, 0 \le r < m \end{matrix}\right\}$$

Thus,

$B_4(T_0{}^m f_0, T_0{}^m f_0, T_0{}^m f_0, T_0{}^q f_0) >$

$max\left\{\begin{matrix} B_4\left(T_0{}^i f_0, T_0{}^i f_0, T_0{}^i f_0, T_0{}^r f_0\right) \\ : 0 \le i \le p, 0 \le r < m \end{matrix}\right\}$ (iv) Suppose, if possible $B_4(T_0{}^m f_0, T_0{}^m f_0, T_0{}^m f_0, T_0{}^q f$

$h\max\left\{\begin{matrix} B_4\left(T_0{}^i f_0, T_0{}^i f_0, T_0{}^i f_0, T_0{}^r f_0\right) \\ : 0 \le i \text{ and } r < m \end{matrix}\right\}$

and so, by (iv),

$B_4\left(T_0{}^m f_0, T_0{}^m f_0, T_0{}^m f_0, T_0^q f_0\right) \le h\max\left\{\begin{matrix} B_4\left(T_0{}^i f_0, T_0{}^i f_0, T_0^i f_0, T_0{}^r f_0\right) \\ : p < i \text{ and } r < m \end{matrix}\right\}$

(v) Using the inequality (5.2), we can write

$B_4(T_0{}^m f_0, T_0{}^m f_0, T_0{}^m f_0, T_0{}^q f_0) \le h^k \max\left\{\begin{matrix} B_4\left(T_0{}^i f_0, T_0{}^i f_0, T_0{}^i f_0, T_0{}^r f_0\right) \\ : p < i \text{ and } r < m \end{matrix}\right\}$,

for $k = 1,2, \ldots$

Since, we can omit the terms of the form as $B_4\left(T_0{}^i f_0, T_0{}^i f_0, T_0{}^i f_0, T_0{}^r f_0\right)$ with

$0 \le i \le p$ 0by (iii)Now, on letting $k \to \infty$ we get

$B_4(T_0{}^m f_0, T_0{}^m f_0, T_0{}^m f_0, T_0{}^q f_0) = 0$, a contradiction (ii).

Therefore, the inequality (iv) does not hold.

Hence $B_4(T_0{}^m f_0, T_0{}^m f_0, T_0{}^m f_0, T_0{}^q f_0) > h\max\left\{\begin{matrix} B_4\left(T_0{}^i f_0, T_0{}^i f_0, T_0{}^i f_0, T_0^q f_0\right) \\ : 0 \le i \text{ and } r < m \end{matrix}\right\}$,

Thus, the sequence $\{T_0{}^n f_0 : n = 1,2, \ldots\}$ is B_4 - bounded.

Now, we put

$$N = \sup\{B_4(T_0{}^r f_0, T_0{}^r f_0, T_0{}^r f_0, T_0^s f_0) : \mathrm{r}, \mathrm{s} = 0,1,2, \ldots\} < \infty$$

Therefore, for $\epsilon > 0$. Choose M, such that, $h^M N < \epsilon$. For $m, n \ge M$ max$\{p, q\}$

and using the inequality (5.2) M times,

we have $B_4(T_0{}^m f_0, T_0{}^m f_0, T_0{}^m f_0, T_0{}^n f_0) \le h^M N < \epsilon$.

Hence, the sequence $\{T_0{}^n f_0 : n = 1,2,\ldots\}$ is a Cauchy sequence in the complete B_4 metric space (Y_0, B_4) and so has a limit u_0 in Y_0.

Therefore $T_0{}^n u \to u$, since T_0 is continuous $T_0(T_0{}^n f_0) \to T_0 u$. That is $T_0{}^{n+1} f_0 \to T_0 u$.

Therefore, $T_0 u = u$

Therefore, u_0 is a fixed point of T_0.

Now we show that u_0 is the one and only one fixed point of T_0.

By the inequality (5.2). $B_4(T_0{}^p f_0, T_0{}^p f_0, T_0{}^p f_0, T_0{}^q g_0)$

$$\leq h\max\left\{\begin{matrix} \mathrm{B}_4(T_0{}^r f_0, T_0{}^r f_0, T_0{}^r f_0, T_0{}^s g_0), \mathrm{B}_4\left(T_0{}^r f_0, T_0{}^r f_0, T_0{}^r f_0, T_0^{r'} g_0\right) \\ \mathrm{B}_4\left(T_0{}^s g_0, T_0{}^s g_0, T_0{}^s g_0, T_0{}^{s'} g_0\right) \\ : 0 \leq r, r' \leq p \text{ and } 0 \leq \mathrm{s}, s' \leq q \end{matrix}\right\}$$

Suppose u_0 and v are two fixed points of T_0.

Putting, $f_0 = u$ and $g_0 = v$ in (5.2), we get $B_4\left(T_0{}^p f_0, T_0{}^p f_0, T_0{}^p f_0, T_0^q g_0\right) <$

$$h\max\left\{\begin{matrix} \mathrm{B}_4(T_0{}^r u, T_0{}^r u, T_0{}^r u, T_0{}^s v), \\ \mathrm{B}_4\left(T_0{}^r u, T_0^r u, T_0{}^r u, T_0^{r'} v\right) \\ \mathrm{B}_4\left(T_0{}^s v, T_0{}^s v, T_0{}^s v, T_0{}^{s'} v\right) \\ : 0 \leq r, r' \leq p \text{ and } 0 \leq \mathrm{s}, s' \leq q \end{matrix}\right\}$$

Therefore, $B_4(u, u, u, v)$

$$\leq h \max\{B_4(u,u,u,v), B_4(u,u,u,u), B_4(u,u,u,v)\} B_4(u,u,u,v) \leq h B_4(u,u,u,v)$$

$\Rightarrow B_4(u, \mathrm{u}, \mathrm{u}, \mathrm{v}) = 0$

Therefore, $\mathrm{u} = \mathrm{v}$.

Therefore, u_0 is one and only one fixed point of T_0 in Y_0.

Corollary 5.1. Let (Y_0, B_4) be a complete B_4 - metric space. Suppose T_0 is a continuous self-map on Y_0, and satisfies the inequality (5.2). $\exists$ positive integers p and q and $0 \leq h < 1$, such that,

$$B_4\left(T_0{}^p f_0, T_0{}^p f_0, T_0{}^p f_0, T_0^q g_0\right) \leq h\max\left\{\begin{matrix} \mathrm{B}_4(T_0{}^r f_0, T_0{}^r f_0, T_0^r f_0, T_0{}^s g_0) \\ \mathrm{B}_4\left(T_0{}^r f_0, T_0^r f_0, T_0^r f_0, T_0{}^{r'} g_0\right) \\ \mathrm{B}_4\left(T_0{}^s g_0, T_0{}^s g_0, T_0{}^s g_0, T_0{}^{s'} g_0\right) \\ : 0 \leq r, r' \leq p \text{ and } 0 \leq \mathrm{s}, s' \leq q \end{matrix}\right\}$$

Then, T_0 has one and only one fixed point in Y_0.

Proof. It is straight forward

Remark 2: The condition that T_0 be continuous when $p, q \geq 2$ is necessary for Theorem 4.3 The following example shows that Theorem 4.4 cannot be always true when T_0 is a discontinuous self-map on Y_0.

Example 5.1. Let $\mathbb{R}$ be the real line. Let us consider the B_4 metric defined in Example 2.2, on $\mathbb{R}$.

Define $T_0: \mathbb{R} \to \mathbb{R}$ by $T_0 f_0 = \begin{cases} 1, & \text{if } f_0 = 0 \\ \frac{f_0}{4}, & \text{if } f_0 \neq 0 \end{cases}$

Then, T_0 is a discontinuous self-map on the complete B_4 - metric space $[0,1]$.

For each $f_0, g_0 \in T_0$.

we

obtain$B_4\left(T_0{}^p f_0, T_0{}^p f_0, T_0{}^p f_0, T_0^q g_0\right) =$

$\frac{1}{4} B_4(T_0{}^{p-1} f_0, T_0{}^{p-1} f_0, T_0{}^{p-1} f_0, T_0{}^{q-1} g_0)$

and so, the inequality $(\mathbf{5.1})$ is satisfied +with $h = \frac{1}{4}$

Conclusions

We presented a new contractive mapping on B_4 - metric spaces, find their relationships with suitable examples and we got, self-map fixed point theorems on B_4- metric spaces. In future we will investigate on B_n - metric spaces.

Acknowledgements

The authors are very grateful to the referee for his/her critical comments.

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