POWER DOMINATOR CHROMATIC NUMBER OF MIDDLE, LINE AND TOTAL GRAPHS OF SUNLET, HELM GRAPHS AND IRREGULAR CHEMICAL CENTRAL GRAPH

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Abstract

Power dominator colouring is a form of vertex colouring for a simple graph G so that each and every vertex in G power dominates a minimum of one colour class. The power dominator chromatic (PDC) number is the least number of colours needed for such colouring in G and will be represented by chi_pd(G). In this article, we determine the power dominator chromatic (PDC) number of line, middle and total graphs of sunlet and helm graphs. Also we obtain the power dominator chromatic (PDC) number of irregular chemical central graph.

AMS Subject Classification: 05C15, 05C69

Keywords: Power dominator chromatic number, Sunlet graph, Helm graph, middle graph, line graph, total graph, irregular chemical central graph.

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1. Introduction

An important and highly explored research area in graph theory with applications in numerous fields is the theory of domination. A graph G = (V, E), is a mathematical structure made up of a finite number of elements, known as vertices, and a finite number of pairings of vertices, known as edges. We take into account that the finite undirected graphs with absence of loops and numerous edges. If every vertex in a subset S of a graph G = (V, E) has not less than one neighbour in S, then S is a dominating set of G. $\gamma(G)$ is The cardinal value of the least dominating set in G and is the dominating number of the graph G. The idea of domination [9, 10] in graphs has many different forms. The complexity of keeping watch over an electrical system by utilising the fewest possible phase measurement units (PMU's) was formulated in graph theoretical terms by Haynes et al. [11], who also created the idea of power domination. However, graph colouring [1] is another area of graph theory that has attracted the most attention. In graph G, proper colouring [1] is the process of allocating distinct colours to the nodes of G while ensuring that there are no identical colours between adjacent vertices. The fewest number of colours needed to colour G appropriately is known as chromatic number of G, designated by $\chi(G)$. The concept of dominator colouring, which permits minimum of one colour class to be dominated by every vertex, was first presented in [4].

Power dominator colouring (PDC) of a graph G is a new idea of colouring that was introduced by K. Sathish Kumar et al. [3] by merging the notions of colouring and power domination. The power dominator colouring [2, 4] of G is a suitable colouring of G in such a way that each point of the vertex collection V power dominates a vertex at a minimum of one colour class. The smallest cardinal number of colours necessary for a power dominator colouring of G is known as the power dominator chromatic number $\chi_{pd}(G)$. In the field of inorganic chemistry, power dominator colouring of central irregular chemical graphs with the molecular structure that is obtained only among the p-block Elements may be found. Atoms are considered to be vertices, covalent bonds are edges, and valence is the degree of vertices. An irregular chemical graph is one whose molecular structure corresponds to elements of nearby atoms with various valencies.

2. Preliminaries

The k-sunlet graph on *2k* nodes is constructed by connecting *k* edges of degree one to the cycle C_k and is indicated by S_k. A Helm graph H_k, $k \geq 3$ is the graph constructed through the wheel graph W_k by including pendant edges at each nodes on the rim of the wheel W_k. For a connected graph, $M(G)$ is used to represent the middle graph of G by the graph with point collection $V(G) \cup E(G)$, in which two nodes are connect one another if (i) they are the neighbouring lines in G or (ii) the first is a node of G, while the second is the line that connects it. The line graph L(G) of a connected graph G is a graph such that (i) each of the vertices in L(G) denotes one of G's edges (ii) two points of L(G) are neighbouring iff their respective edges meet at a common end vertex. T(G) is the total graph of a graph G such that the node set of T corresponding to the edges and points of G and two nodes are adjacent to one another in T iff their respective elements are either neighbouring or incident. By precisely dividing each edge of G once and joining all the non-adjacent vertices of G in C (G), the central graph of G is obtained. A node in a graph G is only adjacent to vertices with distinct degrees, the graph is said to be irregular. Every combination of adjacent vertices on a graph G has a different degree, then the graph is said to be a neighbourly irregular. The molecular structure of any of the relevant element's atoms has a differing valency bond in its nearby atoms, the graph is said to be an irregular chemical graph [6].

3. Power Dominator Chromatic Number Of Line, Middle and Total Graph of Sunlet Graph:

In this section, we present a new result on power dominator chromatic (PDC) number of line, middle and total graph of sunlet graph.

Theorem: 3. 1

The power dominator chromatic (PDC) number of line graph of $S_k, k \geq 3$, is

$$\chi_{pd}(L(S_k)) = \begin{cases} \frac{k}{2}+2 \ , k \ is \ even \\ \left\lceil \frac{k}{2} \right\rceil + 2 \ , k \ is \ odd \end{cases}.$$

Proof:

Consider the sunlet graph S_k with 2k vertices as $V(S_k) = \{v_1, v_2, ..., v_k\} \cup \{v_1^{'}, v_2^{'}, ...v_k^{'}\}$ and 2k number of edges as $E(S_k) = \{e_k\} \cup \{e_l : 1 \leq l \leq k-1\} \cup \{e_l^{'} : 1 \leq l \leq k\}$, e_l is an edge connecting v_l *and* v_{l+1} $(1 \leq l \leq k-1)$ and e_k is between v_1 *and* v_k. Also the edge $e_l^{'}$ is between v_l *and* $v_l^{'}$ $(1 \leq l \leq k)$. According to the concept of line graph $V(L(S_k)) = E(S_k) = \{u_l : 1 \leq l \leq k\} \cup \{u_l^{'} : 1 \leq l \leq k-1\} \cup \{u_k^{'}\}$ where u_k *and* $u_k^{'}$ are vertices corresponding to e_k *and* $e_k^{'}$ respectively. The vertex $u_l^{'}$ $(1 \leq l \leq k-1)$ power dominates u_l *and* u_{l+1}, $u_k^{'}$ power dominates u_l *and* u_1.We assign a spare colour class C_1 for $u_l^{'}$ $(1 \leq l \leq k)$ and either one of the adjacent vertices of $u_l^{'}$ $(1 \leq l \leq k)$ must have a different colour class. The vertices u_l $(1 \leq l \leq k)$ forms a cycle and assign $\frac{k}{2}$ different colours alternatively for u_l $(1 \leq l \leq k)$ while k is even or $\left\lceil \frac{k}{2} \right\rceil$ different colours can be used alternatively for the above while k is odd. Remaining vertices can be assigned by a another spare colour class C_2. This completes the proof.

Theorem: 3.2

The power dominator chromatic (PDC) number of middle graph of $S_k, k \geq 3$, is

$$\chi_{pd}(M(S_k)) = \begin{cases} k+3 \ , k \ is \ even \\ k+4 \ , k \ is \ odd \end{cases}.$$

Proof:

Let the sunlet graph S_k with *2k* vertices as $V(S_k) = \{v_1, v_2, ..., v_k\} \cup \{v_1^{'}, v_2^{'}, ...v_k^{'}\}$ and *2k* number of edges as $E(S_k) = \{e_k\} \cup \{e_l : 1 \leq l \leq k-1\} \cup \{e_l^{'} : 1 \leq l \leq k\}$, e_l is an edge connecting v_l *and* v_{l+1} $(1 \leq l \leq k-1)$ and e_k is between v_1 *and* v_k. Also the edge $e_l^{'}$ is between v_l *and* $v_l^{'}$ $(1 \leq l \leq k)$. Based on the concepts of middle graph, $V(M(S_k)) = \{v_l : 1 \leq l \leq k\} \cup \{v_l^{'} : 1 \leq l \leq k\} \cup \{u_l : 1 \leq l \leq k\} \cup \{u_l^{'} : 1 \leq l \leq k\}$ where

u_l *and* $u_l^{'}$ are vertices corresponding to the edges e_l *and* $e_l^{'}$ respectively. Assign colour class C_1 to the vertices $v_l^{'}\ (1 \le l \le k)$ and the vertices $v_l\ (1 \le l \le k)$. The vertices $v_l^{'}\ (1 \le l \le k)$ power dominates each $u_m^{'}\ (1 \le m \le k)$ *with* $l = m$. Also the vertices $u_l^{'}\ (1 \le l \le k)$ power dominates the remaining nodes in M (S_k). Assign *k* different colours to $u_l^{'}\ (1 \le l \le k)$. The 2k number of vertices v_l and $u_l\ (1 \le l \le k)$ forms a cycle. Each u_l's are adjacent with u_{l-1} *and* u_{l+1}, so two more spare colour classes C_2 and C_3 are required when k is even or one more additional colour C_4 is needed when k is odd. Thus the power dominator chromatic (PDC) number of M (S_k) is *k*+3 when k is even and k+4 when k is odd.

Theorem: 3. 3

The power dominator chromatic number of total graph of $S_k, k \ge 3$, is

$$\chi_{pd}(T(S_k)) = \begin{cases} k+3 \ , k \text{ is even} \\ \\ k+4 \ , k \text{ is odd} \end{cases}.$$

Proof:

Let the sunlet graph S_k with *2k* vertices as $V(S_k) = \{v_1, v_2, ..., v_k\} \cup \{v_1^{'}, v_2^{'}, ...v_k^{'}\}$ and *2k* number of edges as $E(S_k) = \{e_k\} \cup \{e_l : 1 \le l \le k-1\} \cup \{e_l^{'} : 1 \le l \le k\}$, e_l is an edge connecting v_l *and* v_{l+1} $(1 \le l \le k-1)$ and e_k is between v_1 *and* v_k. Also the edge $e_l^{'}$ between v_l *and* $v_l^{'}\ (1 \le l \le k)$. According to the concept of total graph, $V(T(S_k)) = \{v_l : 1 \le l \le k\} \cup \{v_l^{'} : 1 \le l \le k\}$ $\cup \{u_l : 1 \le l \le k\} \cup \{u_l^{'} : 1 \le l \le k\}$ where u_l *and* $u_l^{'}$ are vertices corresponding to the edges e_l *and* $e_l^{'}$ respectively. Also there exists an edge in $T(S_k)$ for every adjacent vertices of S_k. These edges are additional set of edges occur in $T(S_k)$ which are not in M (S_k). Assign colour class C_1 to the nodes $v_l^{'}\ (1 \le l \le k)$ and colour class C_2 for $u_l^{'}\ (1 \le l \le k)$. Assign another *k* different colours to the nodes $v_l\ (1 \le l \le k)$, the vertices still remaining can be coloured by the colour classes C_1 and C_3 when *k* is even or with C_1, C_3 and C_4 when *k* is odd. Thus the result holds.

The power dominator chromatic number (PDC) of total graph of S_6 is presented in Fig 3.1.

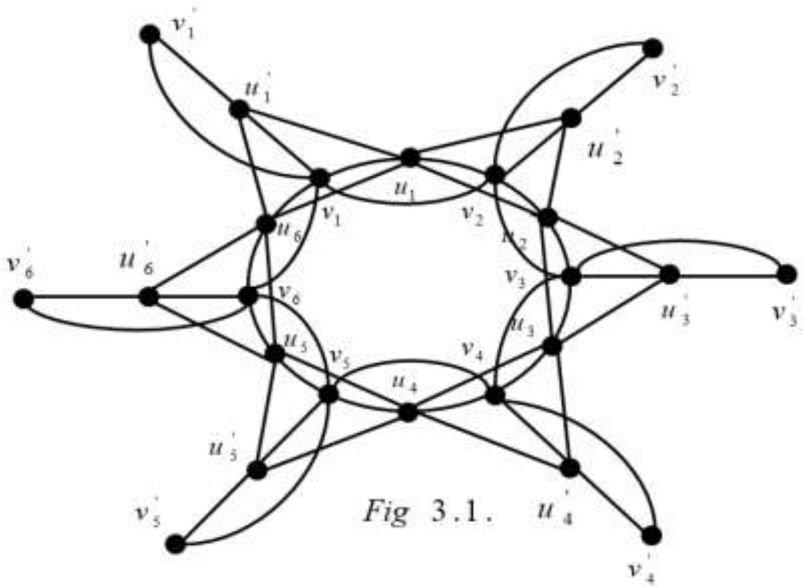

Fig 3.1.

$C_1 = \{v_i' : 1 \le i \le 6 \text{ and } u_1, u_3, u_5\}$; $C_2 = \{u_i' : 1 \le i \le 6\}$; $C_3 = \{u_2, u_4, u_6\}$; $C_4 = \{v_1\}$;
$C_5 = \{v_2\}$; $C_6 = \{v_3\}$; $C_7 = \{v_4\}$; $C_8 = \{v_5\}$; $C_9 = \{v_6\}$;

4. Power Dominator Chromatic Number (PDC) of Line, Middle and Total Graph of Helm Graph:

This section deals a new result on power dominator chromatic (PDC) number of line, middle and total graph of helm graph.

Theorem: 4.1

The power dominator chromatic (PDC) number of line graph of H_k, $k \ge 3$ is,

$$\chi_{pd}(L(H_k)) = \begin{cases} k+3 & \text{when } k \text{ is even} \\ k+4 & \text{when } k \text{ is odd} \end{cases}.$$

Proof:

Let $V(H_k) = \{v_0\} \cup \{v_1, v_2, \ldots, v_k\} \cup \{v_1', v_2', \ldots v_k'\}$ and the edge set $E(H_k) = \{e_l : 1 \le l \le k\} \cup \{e_l' : 1 \le l \le k\} \cup \{f_l : 1 \le l \le k-1\} \cup \{f_k\}$ where e_l is the edge $v_0 v_l$ $(1 \le l \le k)$, e_l' is the edge $v_l v_l'$ $(1 \le l \le k)$, f_l is the edge $v_l v_{l+1}$ $(1 \le l \le k-1)$ and f_k is an edge $v_k v_1$. According to the line graph's definition $V(M(H_k)) = E(H_k)$. The vertices e_l $(1 \le l \le k)$ forms a clique, which power dominates each other. Assign k different colours for e_l $(1 \le l \le k)$. Assign colour class C_1 for e_l' $(1 \le l \le k)$. The nodes f_l $(1 \le l \le k)$ forms a cycle. Either colours C_2 and C_3 or C_2, C_3 and C_4 are needed to make f_l $(1 \le l \le k)$ power dominated. Thus the power dominator chromatic number of $L(H_k)$ is $k+3$ when k is even or $k+4$ when k is odd.

Theorem: 4.2

The power dominator chromatic (PDC) number of middle graph of H_k, $M(H_k)$ is $\chi_{pd}(M(H_k)) = 2k+1$.

Proof:

Consider the helm graph H_k with 2k+1 number of vertices. Let $V(H_k) = \{v_0\} \cup \{v_1, v_2, \ldots, v_k\} \cup \{v_1', v_2', \ldots v_k'\}$ and the edge set $E(H_k) = \{e_l : 1 \le l \le k\} \cup \{e_l' : 1 \le l \le k\} \cup \{f_l : 1 \le l \le k-1\} \cup \{f_k\}$ where e_l is the edge $v_0 v_l$ $(1 \le l \le k)$, e_l' is the edge $v_l v_l'$ $(1 \le l \le k)$, f_l is the edge $v_l v_{l+1}$ $(1 \le l \le k-1)$ and f_k is an edge $v_k v_1$. From the

concept of middle graph, we have $V(M(H_k)) = V(H_k) \bigcup E(H_k)$. Assign colour class C_x for the vertices $v_l\ (1 \le l \le k)$, $v_l^{'}\ (1 \le l \le k)$ and v_0. The vertices $v_l^{'}\ (1 \le l \le k)$ power dominates only the nodes $e_l^{'}\ (1 \le l \le k)$. Either one of the above two must have different colour class. Assign colours $C_l\ (1 \le l \le k)$ for the nodes $e_l^{'}\ (1 \le l \le k)$ which will power dominates all the vertices of $v_l\ (1 \le l \le k)$, $v_l^{'}\ (1 \le l \le k)$, $e_l\ (1 \le l \le k)$ and $f_l\ (1 \le l \le k)$. But the nodes $e_l\ (1 \le l \le k)$ along with v_0 forms a clique. So we have to assign another k colours namely $C_l\ (k+1 \le l \le 2k)$ for the vertices $e_l\ (1 \le l \le k)$. For making the vertex v_0 power dominated, the colour class $C_l\ (k+1 \le l \le 2k-1)$ can be considered as the spare colour classes to colour the vertices $f_l\ (1 \le l \le k)$ without violating the concept of proper colouring. Thus the power dominator chromatic (PDC) number of $M(H_k)$ is *2k+1.*
The power dominator chromatic number of total graph of H_6 is present in Fig 4.1.

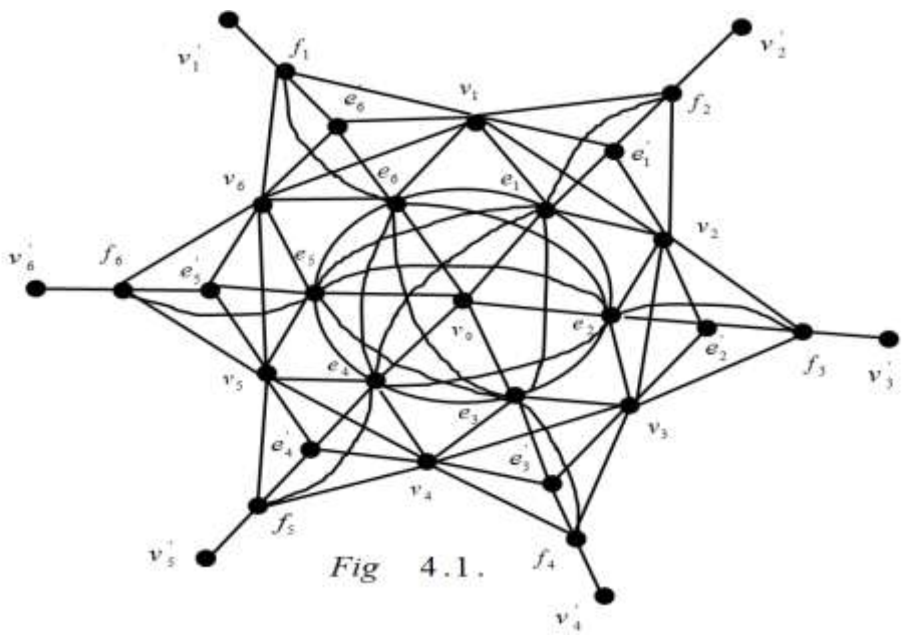

Fig 4.1.

$C_1 = \{e_1^{'}\};\ C_2 = \{e_2^{'}\};\ C_3 = \{e_3^{'}\};\ C_4 = \{e_4^{'}\};\ C_5 = \{e_5^{'}\};\ C_6 = \{e_6^{'}\};\ C_7 = \{e_1\};\ C_8 = \{e_2\};$
$C_9 = \{e_3\};\ C_{10} = \{e_4\};\ C_{11} = \{e_5\};\ C_{12} = \{e_6\};\ C_{13} = \{v_0, v_i : 1 \le i \le 6\ \ and\ \ v_i^{'} : 1 \le i \le 6\};$

Theorem: 4.3

The power dominator chromatic number of total graph of H_k, $k \ge 3$ is $\chi_{pd}(T(H_k)) = 2k+1.$

Proof:

Let the vertex set $V(H_k) = \{v_0\} \bigcup \{v_1, v_2, ..., v_k\} \bigcup \{v_1^{'}, v_2^{'}, ...v_k^{'}\}$ and the edge set $E(H_k) = \{e_l : 1 \le l \le k\} \bigcup \{e_l^{'} : 1 \le l \le k\} \bigcup \{f_l : 1 \le l \le k-1\} \bigcup \{f_k\}$ where e_l is the edge $v_0 v_l\ (1 \le l \le k)$, $e_l^{'}$ is the edge $v_l v_l^{'}\ (1 \le l \le k)$, f_l is an edge $v_l v_{l+1}\ (1 \le l \le k-1)$ and f_k is an edge connecting v_k *and* v_1. According to the definition of total graph $V(T(H_k)) = V(H_k) \bigcup E(H_k)$. Also there exist an edge in $T(H_k)$ for every adjacent vertices of H_k. These edges are additional set of edges occur in $T(H_k)$ not in $M(H_k)$. The vertices v_0 and $e_l\ (1 \le l \le k)$ are adjacent with each other. Also the vertex v_0 adjoint with $v_l\,(1 \le l \le k)$. Assign colour $C_l\,(1 \le l \le k)$ for $v_l\,(1 \le l \le k)$ respectively. Also assign colour class $C_l\,(k+1 \le l \le 2k)$ for the nodes $e_l\,(1 \le l \le k)$ and $e_l^{'}\,(1 \le l \le k)$. Assigning colour class C_x for the vertices $v_l^{'}\,(1 \le l \le k)$, $f_l\,(1 \le l \le k)$ and v_0 will make the graph power dominated. Thus the power dominator chromatic number of $T(H_k)$ is *2k+1.*

5. Power Dominator Chromatic Number Of Irregular Chemical Central Graph:
Proposition: 5.1

The power dominator chromatic number of central graph of the path P_m,

$$\chi_{pd}(C(P_m)) = \begin{cases} m\,, & when\ m\ is\ even \\ 2\left\lceil \frac{m}{2} \right\rceil - 1, & when\ m\ is\ odd \end{cases}$$

Proof:
The central graph of the path P_m contains *2m-1* number of vertices such as $\{v_1, v_2, \ldots\ldots, v_m, u_1, u_2 \quad ,\ldots\ldots, u_{m-1}\}$. The vertices $u_l\,(1 \le l \le m-1)$ power dominates either v_l *or* v_{l+1}. According to the definition of central graph, vertices $v_l\,(1 \le l \le m)$ are adjacent with all the vertices except v_{l-1} and v_{l+1}. The vertex u_1 power dominates $\{v_1, v_2\}$, u_3 power dominates $\{v_3, v_4\}$ and so on. Assign colour class C_1 to the vertices $\{u_1, u_3, u_{5,} \ldots.\}$. Assign Colour class C_2 for $\{v_1, v_2\}$, Colour class C_3 for $\{v_3, v_4\}$ and so on. This process required minimum of $\left\lceil \frac{m}{2} \right\rceil$ new colours. And the remaining vertices $\{u_2, u_4, u_{6,} \ldots.\}$ needs another $\left\lceil \frac{m-2}{2} \right\rceil$ new colours when m is even and $\left\lceil \frac{m-3}{2} \right\rceil$ new colours when m is odd.

The power dominator chromatic number (PDC) of irregular chemical central graph that is central graph of the Carbon Tree of Octane C_8H_{18} is presented in Fig 5.1.

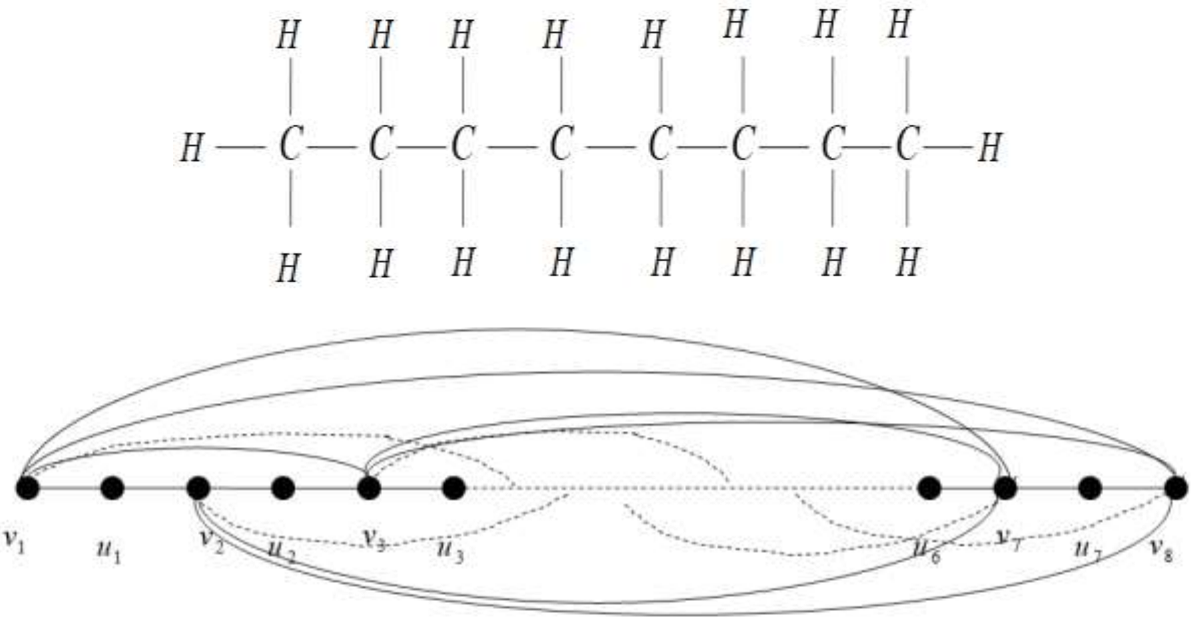

Fig 5.1

$C_1 = \{u_1, u_3, u_5, u_7\}$ $C_2 = \{v_1, v_2\}$ $C_3 = \{u_2\}$ $C_4 = \{v_3, v_4\}$
$C_5 = \{u_4\}$ $C_6 = \{v_5, v_6\}$ $C_7 = \{u_6\}$ $C_8 = \{v_7, v_8\}$.

5. Conclusion

In this article, we have discussed power dominator chromatic (PDC) number for line, middle and total graph of sunlet and helm graph. Also we found the power dominator chromatic (PDC) number of irregular chemical central graph. This study can be expanded to determine the graph families for which the chromatic numbers of the dominator and power dominator are equivalent.

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