Secure Connected Dominating Sets and Secure Connected Domination Polynomials of Fan Graph

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Abstract

Let $G = (V, E)$ be a simple graph. A connected dominating set S of $V(G)$ is a secure connected dominating set of G if for each $u \in V(G)\backslash S$, there exists $v \in S$ such that $uv \in E(G)$ and the set $(S\backslash\{v\}) \cup \{u\}$ is a connected dominating set of G. The minimum cardinality of a secure connected dominating set of G, denoted by $\gamma_{sc}(G)$, is called the secure connected domination number of G. Let F_n be a fan graph with $n + 1$ vertices and let $D_{sc}(F_n, i)$ denote the family of all secure connected dominating sets of F_n with cardinality i. Let $d_{sc}(F_n, i) = |D_{sc}(F_n, i)|$. In this paper, we construct $D_{sc}(F_n, i)$ and obtain the recursive formula for $d_{sc}(F_n, i)$. Using this recursive formula, we construct the polynomial, $(F_n, i) = \sum_{i=\gamma_{sc}(F_n)}^{n+1} D_{sc}(F_n, i)\, x^i$ which we call secure connected domination polynomial of F_n and obtain some properties of this polynomial.

Keywords: Domination, Connected Domination, Secure Connected Domination, Secure Connected Dominating Set, Secure Connected Domination Polynomial.

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INTRODUCTION

Let $G = (V, E)$ be a simple graph. The terms "dominating set" and "domination number" were coined by Oystein Ore in his book on Graph Theory [6]. If the subgraph $< S >$ induced by S is connected in G, then the dominating set S is termed to be a connected dominating set [4]. The concept of secure connected domination in graphs was introduced by Amerkhan G. Cabaro, Sergio S. Canoy, Jr. and Imelda S. Aniversario [1]. A connected dominating set S of $V(G)$ is a secure connected dominating set of G if for each $u \in V(G)\backslash S$, there exists $v \in S$ such that $uv \in E(G)$ and the set $(S\backslash\{v\}) \cup \{u\}$ is a connected dominating set of G. The minimum cardinality of a secure connected dominating set of G, denoted by $\gamma_{sc}(G)$, is called the secure connected domination number of G. Let F_n be a fan graph with $n + 1$ vertices and let $D_{sc}(F_n, i)$ denote the family of all secure connected dominating sets of F_n with cardinality i and $d_{sc}(F_n, i) = |D_{sc}(F_n, i)|$. The polynomial, $D_{sc}(F_n, i) = \sum_{i=\gamma_{sc}(F_n)}^{n+1} D_{sc}(F_n, i)\, x^i$ which we call secure connected domination polynomial of F_n. Throughout this paper, $[n + 1]$ denote the set $\{1, 2, 3, \dots, n, n + 1\}$.

To generalize the secure connected dominating set of F_n, we denote the vertex with degree n as $\{1\}$ and the remaining vertices as $\{2, 3, \dots, n, n+1\}$ respectively.

Example 1.1

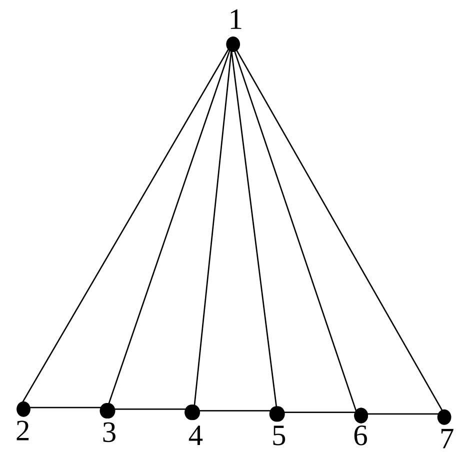

Figure 1: Fan Graph F_6

Here V = {1, 2, 3, 4, 5, 6, 7}; S = {1, 3, 6} is a connected dominating set; V – S = {2, 4, 5, 7}. Also, {1, 2, 6}, {1, 4, 6}, {1, 3, 5} and {1, 3, 7} are connected dominating sets. Therefore, S = {1, 3, 6} is a secure connected dominating set of F_6.

I. SECURE CONNECTED DOMINATING SETS OF FAN GRAPH

Let the family of all secure connected dominating sets of F_n with cardinality i is denoted by $D_{sc}(F_n, i)$. In this section, we will study the secure connected dominating sets of Fan Graph. In order to prove the main result of this article, we need the upcoming lemmas.

Theorem 2.1.[1] *Let F_n be a fan graph of order $n+1$, then* $\gamma_{sc}(F_n) = \begin{cases} 1 & if\ n = 1,2 \\ \left\lceil \frac{n}{3} \right\rceil + 1 & if\ n > 2 \end{cases}$.

Lemma 2.2. *Let F_n be a fan graph with $n+1$ vertices and $D_{sc}(F_n, i)$ be the family of secure connected dominating sets with cardinality i. Then $D_{sc}(F_n, i) = \emptyset$ if and only if $i < \left\lceil \frac{n}{3} \right\rceil + 1$ or $i > n+1$ for all $n > 2$.*

Proof: Since the minimum cardinality of the secure connected dominating set is $\left\lceil \frac{n}{3} \right\rceil + 1$, for all $n > 2$, there cannot exists a set with cardinality less than this minimum cardinality.
Hence, $D_{sc}(F_n, i) = \emptyset$ if $i < \left\lceil \frac{n}{3} \right\rceil + 1$.
Also, since F_n has $n+1$ vertices, there cannot exists a secure connected dominating set with cardinality greater than the number of vertices of the fan graph.
Hence, $D_{sc}(F_n, i) = \emptyset$ if $i > n+1$.

Lemma 2.3. *If a graph G contains a fan graph of order 3k – 2, then for every $k \geq 2$, the secure connected dominating set of G must contain atleast k vertices of the fan graph.*

Proof: Consider a fan graph of order $3k - 2$. By the definition of secure connected dominating set, for every $k \geq 2$, there exists atleast k vertices to dominate the $3k - 2$ vertices of the fan graph. Hence, the proof.

Lemma 2.4. *If $Y \in D_{sc}(F_{n-4}, i-1)$ and there exists $x \in [n+1]$ such that $Y \cup \{x\} \in D_{sc}(F_n, i)$, then $Y \in D_{sc}(F_{n-3}, i-1)$.*

Proof: Suppose that $Y \notin D_{sc}(F_{n-3}, i-1)$.
Since $Y \in D_{sc}(F_{n-4}, i-1)$, then atleast one vertex labelled $n-3$ or $n-4$ is in Y. If $n-3 \in Y$,then $Y \in D_{sc}(F_{n-3}, i-1)$, a contradiction. Hence $n-4 \in Y$, but in this case, $Y \cup \{x\} \notin D_{sc}(F_n, i)$, for any $x \in [n+1]$, also a contradiction.
Therefore, $Y \in D_{sc}(F_{n-3}, i-1)$.

Lemma 2.5. *(i) If* $D_{sc}(F_{n-1}, i-1) = D_{sc}(F_{n-3}, i-1) = \emptyset$, then $D_{sc}(F_{n-2}, i-1) = \emptyset$.
(ii) If $D_{sc}(F_{n-1}, i-1) \neq \emptyset$ and $D_{sc}(F_{n-3}, i-1) \neq \emptyset$, then $D_{sc}(F_{n-2}, i-1) \neq \emptyset$.
(iii) If $D_{sc}(F_{n-1}, i-1) = D_{sc}(F_{n-2}, i-1) = D_{sc}(F_{n-3}, i-1) = \emptyset$, then $D_{sc}(F_n, i) = \emptyset$.

Proof: *(i)* Since $D_{sc}(F_{n-1}, i-1) = D_{sc}(F_{n-3}, i-1) = \emptyset$, by lemma 2.2, $i-1 < \left\lceil \frac{n-1}{3} \right\rceil + 1$ or $i-1 > n$ and $i-1 < \left\lceil \frac{n-3}{3} \right\rceil + 1$ or $i-1 > n-2$. Therefore, $i-1 < \left\lceil \frac{n-3}{3} \right\rceil + 1$ or $i-1 > n$. From this we can write, $i-1 < \left\lceil \frac{n-3}{3} \right\rceil + 1 < \left\lceil \frac{n-2}{3} \right\rceil + 1$ or $i-1 > n > n-1$.
In either case, $D_{sc}(F_{n-2}, i-1) = \emptyset$.

(ii) Suppose that $D_{sc}(F_{n-2}, i-1) = \emptyset$, then by lemma 2.2, $i-1 < \left\lceil \frac{n-2}{3} \right\rceil + 1$ or $i-1 > n-1$.
If $i-1 < \left\lceil \frac{n-2}{3} \right\rceil + 1$, then $i-1 < \left\lceil \frac{n-1}{3} \right\rceil + 1$. Therefore, $D_{sc}(F_{n-1}, i-1) = \emptyset$, a contradiction.
Also, if $i-1 > n-1$, then $i-1 > n-2$. Therefore, $D_{sc}(F_{n-3}, i-1) = \emptyset$, a contradiction.
Hence, $D_{sc}(F_{n-2}, i-1) \neq \emptyset$.

(iii) Suppose that $D_{sc}(F_n, i) \neq \emptyset$.
Let $Y \in D_{sc}(F_n, i)$, then there exists atleast one vertex labelled n or $n+1$ is in Y. If $n+1 \in Y$, then atleast one vertex labelled $n-2$, $n-1$ or n is in Y. If $n-1 \in Y$ or $n \in Y$, then $Y - \{n+1\} \in D_{sc}(F_{n-1}, i-1)$, a contradiction. If $n-2 \in Y$, then $Y - \{n+1\} \in D_{sc}(F_{n-2}, i-1)$, a contradiction. Therefore, $n+1 \notin Y$. Now, if $n \in Y$, then atleast one vertex labelled $n-1$, $n-2$ or $n-3$ is in Y. If $n-2 \in Y$ or $n-3 \in Y$, then $Y - \{n\} \in D_{sc}(F_{n-3}, i-1)$, a contradiction. If $n-1 \in Y$, then $Y - \{n\} \in D_{sc}(F_{n-2}, i-1)$, a contradiction. Therefore, $n \notin Y$. Hence, there does not exist a $Y \in D_{sc}(F_n, i)$ for which $D_{sc}(F_{n-1}, i-1) = D_{sc}(F_{n-2}, i-1) = D_{sc}(F_{n-3}, i-1) = \emptyset$. Therefore, $D_{sc}(F_n, i) \neq \emptyset$.

Lemma 2.6. *If $D_{sc}(F_n, i) \neq \emptyset$, then*

(i) $D_{sc}(F_{n-1}, i-1) \neq \emptyset$, $D_{sc}(F_{n-2}, i-1) = \emptyset$ *and* $D_{sc}(F_{n-3}, i-1) = \emptyset$ *if and only if* $i = n+1$.

(ii) $D_{sc}(F_{n-1}, i-1) = \emptyset$, $D_{sc}(F_{n-2}, i-1) = \emptyset$ *and* $D_{sc}(F_{n-3}, i-1) \neq \emptyset$ *if and only if* $n = 3k$ *and* $i = k+1$, *for all* $k > 1$.

(iii) $D_{sc}(F_{n-1}, i-1) \neq \emptyset$, $D_{sc}(F_{n-2}, i-1) \neq \emptyset$ and $D_{sc}(F_{n-3}, i-1) = \emptyset$ if and only if $i = n$.

(iv) $D_{sc}(F_{n-1}, i-1) = \emptyset$, $D_{sc}(F_{n-2}, i-1) \neq \emptyset$ *and* $D_{sc}(F_{n-3}, i-1) \neq \emptyset$ *if and only if* $n = 3k-1$ *and* $i = \left\lceil \frac{3k-1}{3} \right\rceil + 1$, *for all* $k \geq 2$.

(v) $D_{sc}(F_{n-1}, i-1) \neq \emptyset$, $D_{sc}(F_{n-2}, i-1) \neq \emptyset$ *and* $D_{sc}(F_{n-3}, i-1) \neq \emptyset$ *if and only if* $\left\lceil \frac{n-1}{3} \right\rceil + 2 \leq i \leq n-1$.

Proof: *(i)* (⟹) Since $D_{sc}(F_{n-2}, i-1) = D_{sc}(F_{n-3}, i-1) = \emptyset$, by lemma 2.2, $i-1 < \left\lceil \frac{n-2}{3} \right\rceil + 1$ or $i-1 > n-1$ and $i-1 < \left\lceil \frac{n-3}{3} \right\rceil + 1$ or $i-1 > n-2$. Therefore, $i-1 < \left\lceil \frac{n-3}{3} \right\rceil + 1$ or $i-1 > n-1$.
If $i-1 < \left\lceil \frac{n-3}{3} \right\rceil + 1$, then $i-1 < \left\lceil \frac{n-1}{3} \right\rceil + 1$ and hence $D_{sc}(F_{n-1}, i-1) = \emptyset$, a contradiction.
So, $i-1 > n-1$. Also, since $D_{sc}(F_{n-1}, i-1) \neq \emptyset$, $i-1 \leq n$.
Therefore, $n-1 < i-1 \leq n \Rightarrow n < i \leq n+1$. Hence, $i = n+1$.
(⟹)If $i = n+1$, then
$D_{sc}(F_{n-1}, i-1) = D_{sc}(F_{n-1}, n) \neq \emptyset$,
$D_{sc}(F_{n-2}, i-1) = D_{sc}(F_{n-2}, n) = \emptyset$,
$D_{sc}(F_{n-3}, i-1) = D_{sc}(F_{n-3}, n) = \emptyset$.
Hence, the proof.

(ii) (⟹) Since $D_{sc}(F_{n-1}, i-1) = D_{sc}(F_{n-2}, i-1) = \emptyset$, by lemma 2.2, $i-1 < \left\lceil \frac{n-1}{3} \right\rceil + 1$ or $i-1 > n$ and $i-1 < \left\lceil \frac{n-2}{3} \right\rceil + 1$ or $i-1 > n-1$. Therefore, $i-1 < \left\lceil \frac{n-2}{3} \right\rceil + 1$ or $i-1 > n$.
If $i-1 > n$, then $i > n+1$ and hence $D_{sc}(F_n, i) = \emptyset$, a contradiction. So, $i-1 < \left\lceil \frac{n-2}{3} \right\rceil + 1$ (1)
Also, since $D_{sc}(F_{n-3}, i-1) \neq \emptyset$, $i-1 \geq \left\lceil \frac{n-3}{3} \right\rceil + 1$ (2)
Therefore, $\left\lceil \frac{n-3}{3} \right\rceil + 1 \leq i-1 < \left\lceil \frac{n-2}{3} \right\rceil + 1$ (3)
When $n = 3k$ and $k > 1$, $\left\lceil \frac{n-3}{3} \right\rceil = \frac{n}{3}$ and $\left\lceil \frac{n-2}{3} \right\rceil = \frac{n}{3} + 1$.
Thus, when $n = 3k$, (3) holds good and $i = \frac{n}{3} + 1 = k+1$.
When $n \neq 3k$, $\left\lceil \frac{n-3}{3} \right\rceil + 1 = \left\lceil \frac{n}{3} \right\rceil$ and $\left\lceil \frac{n-2}{3} \right\rceil + 1 = \left\lceil \frac{n}{3} \right\rceil$. Therefore, $\left\lceil \frac{n}{3} \right\rceil \leq i-1 < \left\lceil \frac{n}{3} \right\rceil$ which is not possible. Hence, $n = 3k$ and $k > 1$.
(⟸)If $n = 3k$ and $i = k+1$, for all $k > 1$, then by lemma 2.2,
$\left\lceil \frac{n-1}{3} \right\rceil + 1 = \left\lceil \frac{3k-1}{3} \right\rceil + 1 = k+1 = i > i-1$. Therefore, $i-1 < \left\lceil \frac{n-1}{3} \right\rceil + 1$ which implies $D_{sc}(F_{n-1}, i-1) = \emptyset$.
Similarly, $\left\lceil \frac{n-2}{3} \right\rceil + 1 = \left\lceil \frac{3k-2}{3} \right\rceil + 1 = k+1 = i > i-1$. Therefore, $i-1 < \left\lceil \frac{n-2}{3} \right\rceil + 1$ which implies $D_{sc}(F_{n-2}, i-1) = \emptyset$.
Now, $\left\lceil \frac{n-3}{3} \right\rceil + 1 = \left\lceil \frac{3k-3}{3} \right\rceil + 1 = k = i-1$. Therefore, $\left\lceil \frac{n-3}{3} \right\rceil + 1 = i-1$ which implies $D_{sc}(F_{n-3}, i-1) \neq \emptyset$. Hence, the proof.

(iii) (⟹) Since $D_{sc}(F_{n-3}, i-1) = \emptyset$, by lemma 2.2, $i-1 < \left\lceil \frac{n-3}{3} \right\rceil + 1$ or $i-1 > n-2$. Also, since $D_{sc}(F_{n-2}, i-1) \neq \emptyset$, by lemma 2.2, $i-1 \geq \left\lceil \frac{n-2}{3} \right\rceil + 1$ or $i-1 \leq n-1$.
Therefore, $\left\lceil \frac{n-2}{3} \right\rceil + 1 \leq i-1 \leq n-1$. Hence, $i-1 < \left\lceil \frac{n-3}{3} \right\rceil + 1$ is not possible.
Therefore, $i-1 > n-2 \Rightarrow i > n-1$. But $i-1 \leq n-1 \Rightarrow i \leq n$.

Hence, $n-1<i\leq n\Longrightarrow i=n$.
($\Longleftarrow$) Now, if $i=n$, then
$D_{sc}(F_{n-1},i-1)=D_{sc}(F_{n-1},n-1)\neq\emptyset$,
$D_{sc}(F_{n-2},i-1)=D_{sc}(F_{n-2},n-1)\neq\emptyset$,
$D_{sc}(F_{n-3},i-1)=D_{sc}(F_{n-3},n-1)=\emptyset$. Hence, the proof.

(iv) ($\Longrightarrow$) Since $D_{sc}(F_{n-1},i-1)=\emptyset$, by lemma 2.2, $i-1<\left\lceil\frac{n-1}{3}\right\rceil+1$ or $i-1>n$. If $i-1>n$, then $i-1>n-1>n-2$. Hence, by lemma 2.2, $D_{sc}(F_{n-2},i-1)=D_{sc}(F_{n-3},i-1)=\emptyset$, a contradiction. Therefore, $i-1<\left\lceil\frac{n-1}{3}\right\rceil+1$.
Since $D_{sc}(F_{n-2},i-1)\neq\emptyset$, by lemma 2.2,
$i-1\geq\left\lceil\frac{n-2}{3}\right\rceil+1$ or $i-1\leq n-1\Longrightarrow\left\lceil\frac{n-2}{3}\right\rceil+1\leq i-1\leq n-1$.
Therefore, $\left\lceil\frac{n-2}{3}\right\rceil+1\leq i-1<\left\lceil\frac{n-1}{3}\right\rceil+1$ (4)
Similarly, since $D_{sc}(F_{n-3},i-1)\neq\emptyset$, by lemma 2.2,
$i-1\geq\left\lceil\frac{n-3}{3}\right\rceil+1$ or $i-1\leq n-2\Longrightarrow\left\lceil\frac{n-3}{3}\right\rceil+1\leq i-1\leq n-2$.
Therefore, $\left\lceil\frac{n-3}{3}\right\rceil+1\leq i-1<\left\lceil\frac{n-1}{3}\right\rceil+1$ (5)
When $n=3k-1$, $\left\lceil\frac{3k-3}{3}\right\rceil+1\leq i-1<\left\lceil\frac{3k-1}{3}\right\rceil+2$.
Therefore, (4) and (5) holds good when $n=3k-1$ and $i=\left\lceil\frac{3k-1}{3}\right\rceil+1$, for all $k\geq 2$.
When $n\neq 3k-1$, $\left\lceil\frac{n-2}{3}\right\rceil+1=\left\lceil\frac{n-1}{3}\right\rceil+1$ and $\left\lceil\frac{n-3}{3}\right\rceil+1=\left\lceil\frac{n-1}{3}\right\rceil+1$, which is not possible by (4) and (5). Therefore, the only possibility is $n=3k-1$ and $i=\left\lceil\frac{3k-1}{3}\right\rceil+1$, $k\geq 2$.
($\Longleftarrow$) Assume that $n=3k-1$ and $i=\left\lceil\frac{3k-1}{3}\right\rceil+1$.
$\left\lceil\frac{n-1}{3}\right\rceil+1=\left\lceil\frac{3k-2}{3}\right\rceil+1=\left\lceil\frac{3k-1}{3}\right\rceil+1=i>i-1\Longrightarrow i-1<\left\lceil\frac{n-1}{3}\right\rceil+1\Longrightarrow D_{sc}(F_{n-1},i-1)=\emptyset$.
$\left\lceil\frac{n-2}{3}\right\rceil+1=\left\lceil\frac{3k-3}{3}\right\rceil+1=\left\lceil\frac{3k-1}{3}\right\rceil=i-1\Longrightarrow i-1=\left\lceil\frac{n-2}{3}\right\rceil+1\Longrightarrow D_{sc}(F_{n-2},i-1)\neq\emptyset$.
$\left\lceil\frac{n-3}{3}\right\rceil+1=\left\lceil\frac{3k-4}{3}\right\rceil+1=\left\lceil\frac{3k-1}{3}\right\rceil=i-1\Longrightarrow i-1=\left\lceil\frac{n-3}{3}\right\rceil+1\Longrightarrow D_{sc}(F_{n-3},i-1)\neq\emptyset$.

(v) ($\Longrightarrow$) Since $D_{sc}(F_{n-1},i-1)\neq\emptyset$, by lemma 2.2, $\left\lceil\frac{n-1}{3}\right\rceil+1\leq i-1\leq n$ (6)
Also, since $D_{sc}(F_{n-2},i-1)\neq\emptyset$, by lemma 2.2, $\left\lceil\frac{n-2}{3}\right\rceil+1\leq i-1\leq n-1$ (7)
Also, since $D_{sc}(F_{n-3},i-1)\neq\emptyset$, by lemma 2.2, $\left\lceil\frac{n-3}{3}\right\rceil+1\leq i-1\leq n-2$ (8)
Combining (6), (7) and (8), we get $\left\lceil\frac{n-1}{3}\right\rceil+1\leq i-1\leq n-2\Longrightarrow\left\lceil\frac{n-1}{3}\right\rceil+2\leq i\leq n-1$.
($\Longleftarrow$) Assume that $\left\lceil\frac{n-1}{3}\right\rceil+2\leq i-1\leq n-1$.
From this, we obtain $\left\lceil\frac{n-3}{3}\right\rceil+1\leq i-1\leq n-2$
$\left\lceil\frac{n-2}{3}\right\rceil+1\leq i-1\leq n-1$
$\left\lceil\frac{n-1}{3}\right\rceil+1\leq i-1\leq n$
Therefore, $D_{sc}(F_{n-3},i-1)\neq\emptyset$, $D_{sc}(F_{n-2},i-1)\neq\emptyset$, $D_{sc}(F_{n-1},i-1)\neq\emptyset$.

Theorem 2.7. *For every* $n \geq 4$ *and* $i \geq \left\lceil \frac{n}{3} \right\rceil + 1$

(i) *If* $D_{sc}(F_{n-1}, i-1) \neq \emptyset$, $D_{sc}(F_{n-2}, i-1) = \emptyset$ and $D_{sc}(F_{n-3}, i-1) = \emptyset$, *then* $D_{sc}(F_n, i) = \{[n+1]\}$.

(ii) *If* $D_{sc}(F_{n-1}, i-1) = \emptyset$, $D_{sc}(F_{n-2}, i-1) = \emptyset$ *and* $D_{sc}(F_{n-3}, i-1) \neq \emptyset$, *then* $D_{sc}(F_n, i) = \{1, 3, 6, 9, \ldots, n-6, n-3, n\}$.

(iii) *If* $D_{sc}(F_{n-1}, i-1) \neq \emptyset$, $D_{sc}(F_{n-2}, i-1) \neq \emptyset$ *and* $D_{sc}(F_{n-3}, i-1) = \emptyset$, *then* $D_{sc}(F_n, i) = \{[n+1] - \{x\} / x \in [n+1]\}$.

(iv) *If* $D_{sc}(F_{n-1}, i-1) = \emptyset$, $D_{sc}(F_{n-2}, i-1) \neq \emptyset$ *and* $D_{sc}(F_{n-3}, i-1) \neq \emptyset$, *then*

$$D_{sc}(F_n, i) = \begin{cases} \{1,3,6\} \cup \{X \cup \{n\} / X \in D_{sc}(F_{n-3}, i-1) \backslash \{n-3, n-2\} & if n = 5 \\ \{1,3,6,9,\ldots,n-5,n-2,n+1\} \cup \{X \cup \{n\} / X \in D_{sc}(F_{n-3}, i-1) & if\ n > 5 \end{cases}$$

(v) *If* $D_{sc}(F_{n-1}, i-1) \neq \emptyset$, $D_{sc}(F_{n-2}, i-1) \neq \emptyset$, $D_{sc}(F_{n-3}, i-1) \neq \emptyset$, *then*

For $i = n-1$,

$$D_{sc}(F_n, i) = \left\{ X_1 \cup \begin{cases} \{n\} & if\ n-1\ is\ the\ end\ vertex\ of\ X_1 \\ \{n+1\} & if\ n\ is\ the\ end\ vertex\ of\ X_1 \end{cases} \middle/ X_1 \in D_{sc}(F_{n-1}, i-1) \right\}$$
$$\cup \{X_2 \cup \{n+1\} / X_2 \in D_{sc}(F_{n-2}, i-1) \backslash \{2,3,\ldots,n-1\}\}$$
$$\cup \{X_3 \cup \{n\} / X_3 \in D_{sc}(F_{n-3}, i-1)\}$$

For $i = n-2$,

$$D_{sc}(F_n, i) = \left\{ X_1 \cup \begin{cases} \{n\} & if\ n-1\ is\ the\ end\ vertex\ of\ X_1 \\ \{n+1\} & if\ n\ is\ the\ end\ vertex\ of\ X_1 \end{cases} \middle/ X_1 \in D_{sc}(F_{n-1}, i-1) \right\}$$
$$\cup \{X_2 \cup \{n+1\} / X_2 \in D_{sc}(F_{n-2}, i-1)\}$$
$$\cup \{X_3 \cup \{n\} / X_3 \in D_{sc}(F_{n-3}, i-1) \backslash \{2,3,\ldots,n-2\}\}$$

For $\left\lceil \frac{n-1}{3} \right\rceil + 2 \leq i \leq n-3$,

$$D_{sc}(F_n, i) = \left\{ X_1 \cup \begin{cases} \{n\} & if\ n-1\ is\ the\ end\ vertex\ of\ X_1 \\ \{n+1\} & if\ n\ is\ the\ end\ vertex\ of\ X_1 \end{cases} \middle/ X_1 \in D_{sc}(F_{n-1}, i-1) \right\}$$
$$\cup \{X_2 \cup \{n+1\} / X_2 \in D_{sc}(F_{n-2}, i-1)\}$$
$$\cup \{X_3 \cup \{n\} / X_3 \in D_{sc}(F_{n-3}, i-1)\}$$

Proof: *(i)* Since $D_{sc}(F_{n-2}, i-1) = D_{sc}(F_{n-3}, i-1) = \emptyset$ and $D_{sc}(F_{n-1}, i-1) \neq \emptyset$, by lemma 2.6*(i)*, we have $i = n+1$. Therefore, $D_{sc}(F_n, i) = D_{sc}(F_n, n+1) = \{[n+1]\}$.

(ii) Since $D_{sc}(F_{n-1}, i-1) = D_{sc}(F_{n-2}, i-1) = \emptyset$ and $D_{sc}(F_{n-3}, i-1) \neq \emptyset$, by lemma 2.6*(ii)*, $n = 3k$ and $i = k+1$, for all $k > 1$. Therefore, $i = k+1 = \frac{n}{3} + 1$. Hence, $D_{sc}(F_n, i) = D_{sc}\left(F_n, \frac{n}{3} + 1\right)$.
Clearly, the set $\{1, 3, 6, \ldots, n-6, n-3, n\}$ is a secure connected dominating set with $\frac{n}{3} + 1$ elements. The other sets with cardinality $\frac{n}{3} + 1$ are $\{1, 2, 6, 9, \ldots, n-3, n\}$, $\{1, 2, 7, 10, \ldots, n-5, n-2, n+1\}$, etc. But these sets are not secure connected dominating sets. Therefore, $\{1, 3, 6, \ldots, n-6, n-3, n\}$ is the only secure connected dominating set with cardinality $\frac{n}{3} + 1$.
Hence, $D_{sc}(F_n, i) = \{1, 3, 6, 9, \ldots, n-6, n-3, n\}$.

(iii) Since $D_{sc}(F_{n-1}, i-1) \neq \emptyset$, $D_{sc}(F_{n-2}, i-1) \neq \emptyset$ and $D_{sc}(F_{n-3}, i-1) = \emptyset$, by lemma 2.6*(iii)*, $i = n$. Therefore, $D_{sc}(F_n, i) = D_{sc}(F_n, n) = \{[n+1]\} - \{x\} / x \in [n+1]\}$.

(iv) Since $D_{sc}(F_{n-1}, i-1) = \emptyset$, $D_{sc}(F_{n-2}, i-1) \neq \emptyset$ and $D_{sc}(F_{n-3}, i-1) \neq \emptyset$, by lemma 2.6*(iv)*, $n = 3k-1$ and $i = \left\lceil \frac{3k-1}{3} \right\rceil + 1 = k+1$, for every $k \geq 2$.

Case 1: $k = 2$

Since $X = \{1,3\} \in D_{sc}(F_{3k-3}, k)$, then $X \cup \{3k\} \in D_{sc}(F_{3k-1}, k+1)$. Also, since $D_{sc}(F_{3k-4}, k) \neq \emptyset$, then $X \in D_{sc}(F_{3k-4}, k)$. Then X contains atleast one vertex labelled $3k-4$ or $3k-3$. If $3k-3 \in X$ or $3k-4 \in X$, then $X \cup \{3k-1\} \in D_{sc}(F_{3k-1}, k+1)$. If $\{3k-3, 3k-4\} \in X$, then $X \cup \{3k-1\}$ and $X \cup \{3k\}$ does not belong to $D_{sc}(F_{3k-1}, k+1)$, since $X \cup \{3k-1\}$ and $X \cup \{3k\}$ are not connected dominating sets.

Therefore, $\{1,3,6\} \cup \{X \cup \{3k-1\} / X \in D_{sc}(F_{3k-4}, k) \backslash \{3k-3, 3k-4\}\} \subseteq \backslash D_{sc}(F_{3k-1}, k+1)$ (9)

Now, let $Y \in D_{sc}(F_{3k-1}, k+1)$, then atleast one vertex labelled $3k$ or $3k-1$ is in Y . If $3k \in Y$, then atleast one vertex labelled $3k-1$, $3k-2$ or $3k-3$ is in Y . If $3k-1$ or $3k-2$ is in Y , then $Y - \{3k\} \in D_{sc}(F_{3k-2}, k)$, a contradiction because $D_{sc}(F_{3k-2}, k) = \emptyset$. Hence, $3k-1 \notin Y$, $3k-2 \notin Y$ and $3k-3 \in Y$. Therefore, $Y = X \cup \{3k\}$ for some $X \in D_{sc}(F_{3k-3}, k)$, that is $Y = \{1,3,6\}$. Now, suppose that $3k-1 \in Y$ and $3k \notin Y$, then atleast one vertex labelled $3k-2$, $3k-3$ or $3k-4$ is in Y. If $3k-2 \in Y$, then $Y - \{3k-1\} \in D_{sc}(F_{3k-3}, k) = \{1,3,6\}$, a contradiction because $3k-2 \notin X$, for all $X \in D_{sc}(F_{3k-3}, k)$. Therefore, $3k-3$ or $3k-4$ is in Y, but $3k-2 \notin Y$. Thus, $Y = X \cup \{3k-1\}$, for some $X \in D_{sc}(F_{3k-4}, k)$ except for the set $\{3k-3, 3k-4\} \in X$, because $\{3k-3, 3k-4\} \cup \{3k-1\}$ is not a connected dominating set. Hence, $Y = X \cup \{3k-1\}$, for some $X \in D_{sc}(F_{3k-4}, k) \backslash \{3k-3, 3k-4\}$.

Therefore, $D_{sc}(F_{3k-1}, k+1) \subseteq \{1,3,6\} \cup \{X \cup \{3k-1\} / X \in D_{sc}(F_{3k-4}, k) \backslash \{3k-3, 3k-4\}\}$ (10)

From (9) and (10),we have for $k = 2$,

$$D_{sc}(F_{3k-1}, k+1) = \{1,3,6\} \cup \{X \cup \{3k-1\} / X \in D_{sc}(F_{3k-4}, k) \backslash \{3k-3, 3k-4\}\}.$$

Case 2: $k > 2$

Since $X = \{1,3,6,9,\ldots,3k-6,3k-3\} \in D_{sc}(F_{3k-3}, k)$, then $X \cup \{3k\} \in D_{sc}(F_{3k-1}, k+1)$. Also, since $D_{sc}(F_{3k-4}, k) \neq \emptyset$, then $X \in D_{sc}(F_{3k-4}, k)$. Then X contains atleast one vertex labelled $3k-4$ or $3k-3$. If $3k-3 \in X$ or $3k-4 \in X$, then $X \cup \{3k-1\} \in D_{sc}(F_{3k-1}, k+1)$. Therefore,

$\{1,3,6,9,\ldots,3k-6,3k-3,3k\} \cup \{X \cup \{3k-1\} / X \in D_{sc}(F_{3k-1}, k) \subseteq D_{sc}(F_{3k-1}, k+1)$ (11)

Now, let $Y \in D_{sc}(F_{3k-1}, k+1)$, then atleast one vertex labelled $3k$ or $3k-1$ is in Y. If $3k \in Y$, then atleast one vertex labelled $3k-1$, $3k-2$ or $3k-3$ is in Y . If $3k-1$ or $3k-2$ is in Y , then $Y - \{3k\} \in D_{sc}(F_{3k-2}, k)$, a contradiction because $D_{sc}(F_{3k-2}, k) = \emptyset$. Hence, $3k-1 \notin Y$, $3k-2 \notin Y$ and $3k-3 \in Y$.

Therefore, $Y = X \cup \{3k\}$ for some $X \in D_{sc}(F_{3k-3}, k)$, that is $Y = \{1,3,6,9,\ldots,3k-6,3k-3,3k\}$. Now, suppose that $3k-1 \in Y$ and $3k \notin Y$, then atleast one vertex labelled $3k-2$, $3k-3$ or $3k-4$ is in Y.

If $3k-2 \in Y$, then $Y - \{3k-1\} \in D_{sc}(F_{3k-3}, k) = \{1,3,6,9,\ldots,3k-6,3k-3\}$, a contradiction because $3k-2 \notin X$, for all $X \in D_{sc}(F_{3k-3}, k)$. Therefore, $3k-3$ or $3k-4$ is in Y, but $3k-2 \notin Y$. Thus, $Y = X \cup \{3k-1\}$, for some $X \in D_{sc}(F_{3k-4}, k)$. Therefore,

$D_{sc}(F_{3k-1}, k+1) \subseteq \{1,3,6,9,\ldots,3k-6,3k-3,3k\} \cup \{X \cup \{3k-1\} / X \in D_{sc}(F_{3k-4}, k)\}$ (12)

From (11) and (12),we have for $k > 2$,

$$D_{sc}(F_{3k-1}, k+1) = \{1,3,6,9,\ldots,3k-6,3k-3,3k\} \cup \{X \cup \{3k-1\} / X \in D_{sc}(F_{3k-4}, k)\}$$

(v) Since $D_{sc}(F_{n-1}, i-1) \neq \emptyset$, $D_{sc}(F_{n-2}, i-1) \neq \emptyset$ and $D_{sc}(F_{n-3}, i-1) \neq \emptyset$, by lemma 2.6*(v)*, $\left\lceil \frac{n-1}{3} \right\rceil + 2 \leq i \leq n-1$. Since $D_{sc}(F_{n-1}, i-1) \neq \emptyset$, let $X_1 \in D_{sc}(F_{n-1}, i-1)$, then atleast one vertex labelled n or $n-1$ is in X_1.

If $n \in X_1$, then $X_1 \cup \{n+1\} \in D_{sc}(F_n, i)$ and if $n-1 \in X_1$, then $X_1 \cup \{n\} \in D_{sc}(F_n, i)$.
Hence, $\left\{X_1 \cup \begin{cases} \{n\} & if\ n-1\ is\ the\ end\ vertex\ of\ X_1 \\ \{n+1\} & if\ n\ is\ the\ end\ vertex\ of\ X_1 \end{cases} \Big/ X_1 \in D_{sc}(F_{n-1}, i-1)\right\} \in D_{sc}(F_n, i)$.
Also, since $D_{sc}(F_{n-2}, i-1) \neq \emptyset$, then for $i = n-1$, $D_{sc}(F_{n-2}, i-1) = D_{sc}(F_{n-2}, n-2) = [n-1] - \{x\} = X_2$. Then $X_2 \cup \{n+1\} \in D_{sc}(F_n, i)$, for every $X_2 \in [n-1] - \{x\}$ except for the set $[n-1] - \{1\}$, since $\{[n-1] - \{1\}\} \cup \{n+1\}$ is not a connected dominating set. Therefore, $X_2 \cup \{n+1\} \in D_{sc}(F_n, i)$, where $X_2 \in D_{sc}(F_{n-2}, i-1) \backslash \{2, 3, \ldots, n-1\}$.
Now, for $\left\lceil \frac{n-1}{3} \right\rceil + 2 \leq i \leq n-2$, let $X_2 \in D_{sc}(F_{n-2}, i-1)$, then there exists atleast one vertex labelled $n-1$ or $n-2$ is in X_2. In either case, $X_2 \cup \{n+1\} \in D_{sc}(F_n, i)$, where $X_2 \in D_{sc}(F_{n-2}, i-1)$.
Also, since $D_{sc}(F_{n-3}, i-1) \neq \emptyset$, then for $i = n-1$,
$D_{sc}(F_{n-3}, i-1) = D_{sc}(F_{n-3}, n-2) = [n-2] = X_3$. Then, $X_3 \cup \{n\} \in D_{sc}(F_n, i)$.
Now, for $i = n-2$,
$D_{sc}(F_{n-3}, i-1) = D_{sc}(F_{n-3}, n-3) = [n-2] - \{x\} = X_3$. Then $X_3 \cup \{n\} \in D_{sc}(F_n, i)$, for every $X_3 \in [n-2] - \{x\}$ except for the set $[n-2] - \{1\}$, since $\{[n-2] - \{1\}\} \cup \{n\}$ is not a connected dominating set.
Therefore, $X_3 \cup \{n\} \in D_{sc}(F_n, i)$, where $X_3 \in D_{sc}(F_{n-3}, i-1) \backslash \{2, 3, \ldots, n-2\}$.
Now, for all $\left\lceil \frac{n-1}{3} \right\rceil + 1 < i \leq n-3$, Let $X_3 \in D_{sc}(F_{n-3}, i-1)$, then there exists atleast one vertex labelled $n-3$ or $n-2$ is in X_3. In either case, $X_3 \cup \{n\} \in D_{sc}(F_n, i)$, where $X_3 \in D_{sc}(F_{n-3}, i-1)$.
Hence, For $i = n-1$,

$$\left\{X_1 \cup \begin{cases} \{n\} & if\ n-1\ is\ the\ end\ vertex\ of\ X_1 \\ \{n+1\} & if\ n\ is\ the\ end\ vertex\ of\ X_1 \end{cases} \Big/ X_1 \in D_{sc}(F_{n-1}, i-1)\right\}$$
$$\cup \{X_2 \cup \{n+1\} / X_2 \in D_{sc}(F_{n-2}, i-1) \backslash \{2, 3, \ldots, n-1\}\}$$
$$\cup \{X_3 \cup \{n\} / X_3 \in D_{sc}(F_{n-3}, i-1)\} \subseteq D_{sc}(F_n, i)$$

For $i = n-2$,

$$\left\{X_1 \cup \begin{cases} \{n\} & if\ n-1\ is\ the\ end\ vertex\ of\ X_1 \\ \{n+1\} & if\ n\ is\ the\ end\ vertex\ of\ X_1 \end{cases} \Big/ X_1 \in D_{sc}(F_{n-1}, i-1)\right\}$$
$$\cup \{X_2 \cup \{n+1\} / X_2 \in D_{sc}(F_{n-2}, i-1)\}$$
$$\cup \{X_3 \cup \{n\} / X_3 \in D_{sc}(F_{n-3}, i-1) \backslash \{2, 3, \ldots, n-2\}\} \subseteq D_{sc}(F_n, i)$$

For $\left\lceil \frac{n-1}{3} \right\rceil + 2 \leq i \leq n-3$,

$$\left\{X_1 \cup \begin{cases} \{n\} & if\ n-1\ is\ the\ end\ vertex\ of\ X_1 \\ \{n+1\} & if\ n\ is\ the\ end\ vertex\ of\ X_1 \end{cases} \Big/ X_1 \in D_{sc}(F_{n-1}, i-1)\right\}$$
$$\cup \{X_2 \cup \{n+1\} / X_2 \in D_{sc}(F_{n-2}, i-1)\}$$
$$\cup \{X_3 \cup \{n\} / X_3 \in D_{sc}(F_{n-3}, i-1)\} \subseteq D_{sc}(F_n, i)$$

Now, let $Z \in D_{sc}(F_n, i)$, then there exists atleast one vertex labelled n or $n+1$ is in Z.
If $n+1 \in Z$, then atleast one vertex labelled $n, n-1$ or $n-2$ is in Z. If $n \in Z$, then $Z = Y \cup \{n+1\}$, for some $Y \in D_{sc}(F_{n-1}, i-1)$.
If $n-1 \in Z$ or $n-2 \in Z$, then $Z = Y \cup \{n+1\}$, for some $Y \in D_{sc}(F_{n-2}, i-1) \backslash \{2, 3, \ldots, n-1\}$.
Since $Z = Y \cup \{n+1\}$ such that $Y = \{2, 3, \ldots, n-1\} \in D_{sc}(F_{n-2}, i-1)$ is not a connected dominating set. Hence, $Z \neq Y \cup \{n+1\}$ for $Y = \{2, 3, \ldots, n-1\} \in D_{sc}(F_{n-2}, i-1)$. Therefore, $Z = Y \cup \{n+1\}$, for some $Y \in D_{sc}(F_{n-2}, i-1) \backslash \{2, 3, \ldots, n-1\}$. This happens only when $i = n-1$.
For all other cases, $Y = Z \cup \{n+1\}$, where $Y \in D_{sc}(F_{n-2}, i-1)$.
Now, if $n \in Z$, then atleast one vertex labelled $n-1, n-2$ or $n-3$ is in Z.

If $n-1 \in Z$, then $Z = Y \cup \{n\}$, for some $Y \in D_{sc}(F_{n-1}, i-1)$.
If $n-2 \in Z$ or $n-3 \in Z$, then $Z = Y \cup \{n\}$, for some $Y \in D_{sc}(F_{n-3}, i-1)\backslash\{2,3,\dots,n-2\}$.
Since $Z = Y \cup \{n\}$, such that $Y = \{2,3,\dots,n-2\} \in D_{sc}(F_{n-3}, i-1)$ is not a connected dominating set. Hence, $Z \neq Y \cup \{n\}$, for $Y = \{2,3,\dots,n-2\} \in D_{sc}(F_{n-3}, i-1)$. Therefore, $Z = Y \cup \{n\}$, for some $Y \in D_{sc}(F_{n-3}, i-1)\backslash\{2,3,\dots,n-2\}$. This happens only when $i = n-2$.
For all the other cases, $Z = Y \cup \{n\}$, where $Y \in D_{sc}(F_{n-3}, i-1)$.
Hence, For $i = n-1$,

$$D_{sc}(F_n, i) \subseteq \left\{ X_1 \cup \begin{cases} \{n\} & if\ n-1\ is\ the\ end\ vertex\ of\ X_1 \\ \{n+1\} & if\ n\ is\ the\ end\ vertex\ of\ X_1 \end{cases} \Big/ X_1 \in D_{sc}(F_{n-1}, i-1) \right\}$$
$$\cup \{X_2 \cup \{n+1\} / X_2 \in D_{sc}(F_{n-2}, i-1)\backslash\{2,3,\dots,n-1\}\}$$
$$\cup \{X_3 \cup \{n\} / X_3 \in D_{sc}(F_{n-3}, i-1)\}$$

For $i = n-2$,

$$D_{sc}(F_n, i) \subseteq \left\{ X_1 \cup \begin{cases} \{n\} & if\ n-1\ is\ the\ end\ vertex\ of\ X_1 \\ \{n+1\} & if\ n\ is\ the\ end\ vertex\ of\ X_1 \end{cases} \Big/ X_1 \in D_{sc}(F_{n-1}, i-1) \right\}$$
$$\cup \{X_2 \cup \{n+1\} / X_2 \in D_{sc}(F_{n-2}, i-1)\}$$
$$\cup \{X_3 \cup \{n\} / X_3 \in D_{sc}(F_{n-3}, i-1)\backslash\{2,3,\dots,n-2\}\}$$

For $\left\lceil \frac{n-1}{3} \right\rceil + 2 \leq i \leq n-3$,

$$D_{sc}(F_n, i) \subseteq \left\{ X_1 \cup \begin{cases} \{n\} & if\ n-1\ is\ the\ end\ vertex\ of\ X_1 \\ \{n+1\} & if\ n\ is\ the\ end\ vertex\ of\ X_1 \end{cases} \Big/ X_1 \in D_{sc}(F_{n-1}, i-1) \right\}$$
$$\cup \{X_2 \cup \{n+1\} / X_2 \in D_{sc}(F_{n-2}, i-1)\}$$
$$\cup \{X_3 \cup \{n\} / X_3 \in D_{sc}(F_{n-3}, i-1)\}$$

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
n															
1	2	1													
2	3	3	1												
3	0	1	4	1											
4	0	0	4	5	1										
5	0	0	3	8	6	1									
6	0	0	1	10	13	7	1								
7	0	0	0	8	22	19	8	1							
8	0	0	0	4	26	40	26	9	1						
9	0	0	0	1	22	61	65	34	10	1					
10	0	0	0	0	13	70	120	98	43	11	1				
11	0	0	0	0	5	61	171	211	140	53	12	1			
12	0	0	0	0	1	40	192	356	343	192	64	13	1		
13	0	0	0	0	0	19	171	483	665	526	255	76	14	1	
14	0	0	0	0	0	6	120	534	1050	1148	771	330	89	15	1

Table 1: $d_{sc}(F_n, i)$, the number of secure connected dominating sets of F_n with cardinality i.

II. SECURE CONNECTED DOMINATION POLYNOMIALS OF FAN GRAPH

In this section, we derive the expression for the secure connected domination polynomial of fan graph F_n, given by $D_{sc}(F_n, i) = \sum_{i=\gamma_{sc}(F_n)}^{n+1} D_{sc}(F_n, i)\, x^i$.

Theorem 3.1. *For* $n \leq 2$,

$D_{sc}(F_1, x) = x^2 + 2x,$

$D_{sc}(F_2, x) = x^3 + 3x^2 + 3x.$

Proof: From the definiton of secure connected domination polynomial of fan graph, it follows.

Theorem 3.2. *(i) For every* $n > 3$, *If* $D_{sc}(F_n, i)$ *is the family of all secure connected dominating sets with cardinality* i *of* F_n, *then*

For $i = n-1$, $|D_{sc}(F_n, i)| = |D_{sc}(F_{n-1}, i-1)| + |D_{sc}(F_{n-2}, i-1)| - 1 + |D_{sc}(F_{n-3}, i-1)|$

For $i = n-2$, $|D_{sc}(F_n, i)| = |D_{sc}(F_{n-1}, i-1)| + |D_{sc}(F_{n-2}, i-1)| + |D_{sc}(F_{n-3}, i-1)| - 1$

Otherwise ($\left\lceil \frac{n}{3} \right\rceil + 1 \leq i \leq n-3$ *and* $i = n, n+1$),

$|D_{sc}(F_n, i)| = |D_{sc}(F_{n-1}, i-1)| + |D_{sc}(F_{n-2}, i-1)| + |D_{sc}(F_{n-3}, i-1)|.$

(ii) For every $n > 5$, $D_{sc}(F_n, x) = x[D_{sc}(F_{n-1}, x) + D_{sc}(F_{n-2}, x) + D_{sc}(F_{n-3}, x)] - (x^{n-2} + x^{n-1})$

with the initial condition $D_{sc}(F_3, x) = x^4 + 4x^3 + x^2$, $D_{sc}(F_4, x) = x^5 + 5x^4 + 4x^3$ *and* $D_{sc}(F_5, x) = x^6 + 6x^5 + 8x^4 + 3x^3$.

Proof: *(i) Case 1*: Suppose that $D_{sc}(F_{n-1}, i-1) \neq \emptyset$, $D_{sc}(F_{n-2}, i-1) = \emptyset$ and $D_{sc}(F_{n-3}, i-1) = \emptyset$, then by lemma 2.6*(i)*, $i = n+1$ and also by theorem 2.7*(i)*, $D_{sc}(F_n, i) = \{[n+1]\}$.
Therefore, $|D_{sc}(F_n, i)| = 1$. In this case, $D_{sc}(F_{n-1}, i-1) = \{[n]\}$. Hence, $|D_{sc}(F_{n-1}, i-1)| = 1$ and $|D_{sc}(F_{n-2}, i-1)| = |D_{sc}(F_{n-3}, i-1)| = 0$. Therefore, for $i = n+1$ the theorem holds.

Case 2: Suppose that $D_{sc}(F_{n-1}, i-1) = \emptyset$, $D_{sc}(F_{n-2}, i-1) = \emptyset$ and $D_{sc}(F_{n-3}, i-1) \neq \emptyset$, then by lemma 2.6*(ii)*, $n = 3k$ and $i = k+1$, $k \geq 1$ and also by theorem 2.7*(ii)*,
$D_{sc}(F_n, i) = \{1, 3, 6, 9, \dots, n-6, n-3, n\}$. Therefore, $|D_{sc}(F_n, i)| = 1$.
In this case, $D_{sc}(F_{n-3}, i-1) = \{1, 3, 6, 9, \dots, n-6, n-3\}$.
Hence, $|D_{sc}(F_{n-3}, i-1)| = 1$ and $|D_{sc}(F_{n-2}, i-1)| = |D_{sc}(F_{n-1}, i-1)| = 0$.
Therefore, in this case the theorem holds.

Case 3: Suppose that $D_{sc}(F_{n-1}, i-1) \neq \emptyset$, $D_{sc}(F_{n-2}, i-1) \neq \emptyset$ and $D_{sc}(F_{n-3}, i-1) = \emptyset$, then by lemma 2.6*(iii)*, $i = n$ and also by theorem 2.7*(iii)*, $D_{sc}(F_n, i) = \{[n+1] - \{x\}/x \in [n+1]\}$.
Therefore, $|D_{sc}(F_n, i)| = n+1$.
In this case, $D_{sc}(F_{n-1}, i-1) = \{[n] - \{x\}/x \in [n]\}$ and $D_{sc}(F_{n-2}, i-1) = \{[n-1]\}$.
Hence, $|D_{sc}(F_{n-1}, i-1)| = n$, $|D_{sc}(F_{n-2}, i-1)| = 1$ and $|D_{sc}(F_{n-3}, i-1)| = 0$.
Therefore, for $i = n$ the theorem holds.

Case 4: Suppose that $D_{sc}(F_{n-1}, i-1) = \emptyset$, $D_{sc}(F_{n-2}, i-1) \neq \emptyset$ and $D_{sc}(F_{n-3}, i-1) \neq \emptyset$, then by lemma 2.6*(iv)*, $n = 3k-1$ and $i = \left\lceil \frac{3k-3}{3} \right\rceil + 1$, $k \geq 2$ and also by theorem 2.7*(iv)*,

$$D_{sc}(F_n, i) = \begin{cases} \{1,3,6\} \cup \{X \cup \{n\}/X \in D_{sc}(F_{n-3}, i-1) \backslash \{n-3, n-2\} & if n = 5 \\ \{1,3,6,9,\ldots,n-5,n-2,n+1\} \cup \{X \cup \{n\}/X \in D_{sc}(F_{n-3}, i-1) & if\ n > 5 \end{cases}$$

If $n = 5$, then $k = 2$ and $i = 3 \Longrightarrow i = n-2$.
Therefore, for $i = n-2$, $|D_{sc}(F_n, i)| = 1 + |D_{sc}(F_{n-3}, i-1)| - 1$. In this case, $|D_{sc}(F_{n-1}, i-1)| = 0$ and $|D_{sc}(F_{n-2}, i-1)| = 1$.
Now, if $n > 5$, then $k > 2 \Longrightarrow i < n-2$. Hence, $|D_{sc}(F_n, i)| = 1 + |D_{sc}(F_{n-3}, i-1)|$. In this case, $|D_{sc}(F_{n-1}, i-1)| = 0$ and $|D_{sc}(F_{n-2}, i-1)| = 1$. Therefore, in this case the theorem holds.

Case 5: Suppose that $D_{sc}(F_{n-1}, i-1) \neq \emptyset$, $D_{sc}(F_{n-2}, i-1) \neq \emptyset$ and $D_{sc}(F_{n-3}, i-1) \neq \emptyset$, then by lemma 2.6 *(v)*, $\left\lceil \frac{n-1}{3} \right\rceil + 2 \leq i \leq n-1$ and also by theorem 2.7 *(v)*,
For $i = n-1$,

$$D_{sc}(F_n, i) = \left\{ X_1 \cup \begin{cases} \{n\} & if\ n-1\ is\ the\ end\ vertex\ of\ X_1 \\ \{n+1\} & if\ n\ is\ the\ end\ vertex\ of\ X_1 \end{cases} \middle/ X_1 \in D_{sc}(F_{n-1}, i-1) \right\}$$
$$\cup \{X_2 \cup \{n+1\}/X_2 \in D_{sc}(F_{n-2}, i-1) \backslash \{2,3,\ldots,n-1\}\}$$
$$\cup \{X_3 \cup \{n\}/X_3 \in D_{sc}(F_{n-3}, i-1)\}$$

Hence, $|D_{sc}(F_n, i)| = |D_{sc}(F_{n-1}, i-1)| + |D_{sc}(F_{n-2}, i-1)| - 1 + |D_{sc}(F_{n-3}, i-1)|$

For $i = n-2$,

$$D_{sc}(F_n, i) = \left\{ X_1 \cup \begin{cases} \{n\} & if\ n-1\ is\ the\ end\ vertex\ of\ X_1 \\ \{n+1\} & if\ n\ is\ the\ end\ vertex\ of\ X_1 \end{cases} \middle/ X_1 \in D_{sc}(F_{n-1}, i-1) \right\}$$
$$\cup \{X_2 \cup \{n+1\}/X_2 \in D_{sc}(F_{n-2}, i-1)\}$$
$$\cup \{X_3 \cup \{n\}/X_3 \in D_{sc}(F_{n-3}, i-1) \backslash \{2,3,\ldots,n-2\}\}$$

Hence, $|D_{sc}(F_n, i)| = |D_{sc}(F_{n-1}, i-1)| + |D_{sc}(F_{n-2}, i-1)| + |D_{sc}(F_{n-3}, i-1)| - 1$

For $\left\lceil \frac{n-1}{3} \right\rceil + 2 \leq i \leq n-3$,

$$D_{sc}(F_n, i) = \left\{ X_1 \cup \begin{cases} \{n\} & if\ n-1\ is\ the\ end\ vertex\ of\ X_1 \\ \{n+1\} & if\ n\ is\ the\ end\ vertex\ of\ X_1 \end{cases} \middle/ X_1 \in D_{sc}(F_{n-1}, i-1) \right\}$$
$$\cup \{X_2 \cup \{n+1\}/X_2 \in D_{sc}(F_{n-2}, i-1)\}$$
$$\cup \{X_3 \cup \{n\}/X_3 \in D_{sc}(F_{n-3}, i-1)\}$$

Hence, $|D_{sc}(F_n, i)| = |D_{sc}(F_{n-1}, i-1)| + |D_{sc}(F_{n-2}, i-1)| + |D_{sc}(F_{n-3}, i-1)|$
Therefore, in this case the theorem holds.

(ii) From the definition of the secure connected domination polynomial of fan graph, it follows.

Theorem 3.3. *The coefficients of $D_{sc}(F_n, x)$ possess the following characteristics:*

(i) $d_{sc}(F_n, n+1) = 1$, *for every* $n \in N$.

(ii) $d_{sc}(F_n, n) = n+1$, *for every* $n \in N$.

(iii) $d_{sc}(F_n, n-1) = \frac{n(n-1)}{2} - 2$, *for every* $n \geq 2$.

(iv) $d_{sc}(F_n, n-2) = \frac{(n-4)(n-3)(n+4)}{6}$, *for every* $n \geq 5$.

(v) $d_{sc}(F_n, n-3) = \frac{(n-4)(n^3-n^2-42n+96)}{24}$, *for every* $n \geq 6$.

(vi) $d_{sc}(F_{3n}, n+1) = 1$, *for every* $n \in N$.

(vii) $d_{sc}(F_{3n}, n+2) = \frac{n(n+3)(n+24)}{162}$, *for every* $n \geq 2$.

(viii) $d_{sc}(F_{3n-1}, n+1) = n+1$, *for every* $n \geq 2$.

Proof: It follows from the table of the secure connected dominating sets of fan graph F_n.

III. CONCLUSION

In this paper, we have studied and discussed some properties of secure connected dominating sets and secure connected domination polynomials of fan graph. We can further study this property for various types of graphs.

REFERENCES

[1] Amerkhan G. Cabaro, Sergio S. Canoy, Jr. and Imelda S. Aniversario, Secure Connected Domination in a Graph, International Journal of Mathematical Analysis, 8, 2065-2074, 2014.

[2] G. Chartrand and P. Zhang, Introduction to Graph Theory, McGraw-Hill, Boston, Mass, USA, 2005.

[3] T.W.Hayles, S.T.Hedetniemi and P.J.Slater, Fundamentals of Domination in Graphs, Vol.208 of Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, New York, NY, USA, 1998.

[4] S.T.Hedetniemi, R.C. and Laskar, Connected Domination in Graphs, In B. Bollob'as, editor, Graph Theory and Combinatorics, Academic Press, London, 1984.

[5] K. Lal Gipson and T. Subha, Secure Dominating Sets and Secure Domination Polynomials of Cycles, Advances and Applications in Discrete Mathematics, 29, 59-83, 2021.

[6] O. Ore, Theory of Graphs, Amer. Math. Soc. Colloq. Publ., 38(Amer. Math. Soc., Providence, RI), 1962.

[7] E. Sampathkumar and H.B.Walikar, The Connected Domination number of a Graph, J. Math. Phys. Sci., 13:607-613, 1979.

[8] Teresa W Haynes, Stephen T Hedetniemi and Peter J Slater, Domination in Graphs, Advanced Topics, Marcel Dekker, 1998.