An existence results of second order fractional neutral integro-differential inclusions with state-dependent delay of order in Banach spaces

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This manuscript is generally concerns with the existence of mild solutions of second order fractional neutral integro-differential inclusions with state-dependent delay in Banach spaces. Applying the fixed point theorem for multi-valued operators resulting from Dhage, we set up existence result with strongly continuous α-order fractional cosine family.

Keywords: fractional Neutral integro-differential inclusions, α-order cosine family, fixed point theorem for multi-valued operators.

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1 Introduction

In several branches of science, mathematical models of physical processes involve differential equations and nowadays, it is proved that most of these models could be better described by fractional order equations (or inclusions) as a result of the material and hereditary properties. Therefore, a great number of applications of the fractional order differential systems exist (see [4, 6, 8, 20, 25]).

In contrast to ordinary differential equations, delay differential equations permit the inclusion of historical actions into mathematical models. A delay differential equation with discrete delay is often presented in the type

$$\dot{x}(t) = f(t, x(t), x(t-\tau))$$

with $f: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Depending on the complexity of the problem, the delay τ may be a constant value $(\tau \geq 0)$, a function of the time $(\tau(t) \geq 0)$, or a function of the solution x itself $(\tau(x(t)) \geq 0)$. Accordingly, equation (1.1) is called a differential equation with constant delay, time-dependent delay, or state-dependent delay, respectively. For more details on this theory and its applications, we suggest the reader to refer [5, 9, 10]. When the right-hand side

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of the problem depends not only on the history of the solution x, but also on the history of the derivative $\dot{x}$, that is,

$$\dot{x}(t) = g(t, x(t), x(t-\tau), \dot{x}(t-\tau)),$$

we have a neutral delay differential equation or neutral functional differential equation [4, 7, 16].

Fractional differential equations with state-dependent delay appear usually in uses as types of equations and due to this motive the analysis of this form of equation has acquired excellent interest in current decades. For the idea of differential equations with state-dependent delay and their uses, we direct the reader to the handbook by Canada et al. [11] and the papers [4, 7, 16].

The cosine function concept is associated with abstract linear second order differential equations. For primary ideas and uses of this concept, we suggest the reader to refer Fattorini [14] and Travis and Webb [23]. The existence, controllability and other qualitative and quantitative components of fractional differential systems are the most developing area of pursuit, in particular, see [3, 21, 24]. In [3], Santos et al. studied the existence of solutions for fractional integro-differential equations with unbounded delay in Banach spaces. Shu and Wang et al. [21] considered the existence results for fractional differential equations with nonlocal conditions of order $\alpha \in (1,2)$. However, existence results for fractional neutral integro-differential inclusions with state-dependent delay in $\mathcal{B}$ phase space adages have not yet been fully investigated.

Motivated by the above mentioned works, in this manuscript, we are concerned with the existence of mild solutions for fractional neutral differential inclusions with state-dependent delay of the form

$$^{C}D_t^{\alpha}[x(t) + U(t, x_t)] \in A[x(t) + U(t, x_t)] + V(t, x_{\rho(t,x_t)}), \quad t \in J = [0, T], \tag{1.1}$$

$$x(t) = \phi(t), \qquad t \in (-\infty, 0], \quad x'(0) = y_0 \in X, \tag{1.2}$$

where $^{C}D_t^{\alpha}$ is the Caputo's fractional derivatives of order $\alpha \in (1,2]$. A is the infinitesimal generator of a strongly continuous α-order cosine family $\{C_\alpha(t)\}_{t\geq 0}$ on X. The function $U: J \times \mathcal{B} \to X$ is a continuous function and $V: J \times \mathcal{B} \to \mathcal{P}(X)$ is a multi-valued map. For any function x described on $(-\infty, T]$ and any $t \in J$, we represent by x_t the element of $\mathcal{B}$ described by $x_t(\theta) = x(t+\theta), \theta \in (-\infty, 0]$. Here x_t represents the history of the state up to the current time t and $\rho: J \times \mathcal{B} \to (-\infty, \infty)$ is an apposite function. $\mathcal{B}$ is the theoretical phase space axioms characterized in Section 2.

Further, we additionally consider the subsequent fractional neutral integro-differential inclusions with state-dependent delay of the form

$$\begin{aligned} ^{C}D_t^{\alpha}[x(t) + U(t, x_t, \int_0^t k_1(t, s, x_s)ds)] &\in A[x(t) + U(t, x_t, \int_0^t k_1(t, s, x_s)ds)] \\ &+ V(t, x_{\rho(t,x_t)}, \int_0^t k_2\big(t, s, x_{\rho(s,x_s)}\big)ds), \quad t \in J = [0, T], \end{aligned} \tag{1.3}$$

$$x(0) = \phi(t), \qquad t \in (-\infty, 0], \quad x'(0) = y_0 \in X, \tag{1.4}$$

where ${}^{C}D_t^{\alpha}, A, \mathcal{B}, \rho, \phi$ and y_0 are same as defined in (1.1) - (1.2). Further $k_i: J \times J \times \mathcal{B} \to X$, (for $i = 1,2$) and $U: J \times \mathcal{B} \times X \to X$ are continuous functions and $V: J \times \mathcal{B} \times X \to \mathcal{P}(X)$ is a multi-valued map.

This manuscript is prepared as follows. In second part, we recall several results, definitions and lemmas that are to be utilized later on to prove our main results. In third part, the main results are based on Dhage's fixed point theorem.

2 Preliminaries

In this part, we remember some basic definitions, lemmas and notations which will be utilized all through this manuscript.

Let X be a Banach space. By $C(J, X)$ we denote the Banach space of continuous functions from J into X with norm

$$\| x \| = \sup\{|x|: t \in J\}.$$

$B(X)$ denotes the Banach space of all bounded linear operators from X into X, with the norm

$$\| N \|_{B(X)} = \sup\{|N(x)|: |x| = 1\}.$$

$L^1(J, X)$ denotes the Banach space of measurable functions $x: J \to X$ which are Bochner integrable, normed by

$$\| x \|_{L^1} = \int_0^T |x(t)| dt.$$

$L^{\infty}(J, X)$ denotes the Banach space of measurable functions $x: J \to X$ which are bounded equipped with the norm

$$\| x \|_{L^{\infty}} = \inf\{d > 0: \| x(t) \| < d, \quad \text{a.e} \quad t \in J\}.$$

We denote the notation $\mathcal{P}(X)$is the family of all subsets of X. Next, we denote the subsequent notations:

$$\mathcal{P}_{cl}(X) = \{y \in \mathcal{P}(X): y \text{ is closed}\},$$
$$\mathcal{P}_{bd}(X) = \{y \in \mathcal{P}(X): y \text{ is bounded}\},$$
$$\mathcal{P}_{cv}(X) = \{y \in \mathcal{P}(X): y \text{ is convex}\},$$
$$\mathcal{P}_{cp}(X) = \{y \in \mathcal{P}(X): y \text{ is compact}\}.$$

The definitions of multi-valued analysis like, convexity, bounded, upper semi-continuous, completely continuous and closed graph theorem are well-known results [15, 12, 17], for this reason, we omit here.

Here, we will utilize a common axioms for the phase space $\mathcal{B}$ which is identical to those presented by Hale and Kato. In particular, $\mathcal{B}$ will be a linear space of function mapping $(-\infty, 0]$ into X endowed with a semi norm $\|\cdot\|_{\mathcal{B}}$ and fulfills the next conditions:

(i) $x \in (-\infty, T] \to X$ is continuous on J and $x_0 \in \mathcal{B}$, then $x_t \in \mathcal{B}$ and x is continuous in $t \in J$ and

$$\| x(t) \| \leq \mathbb{L} \| x_t \|_{\mathcal{B}}$$

where $\mathbb{L} \geq 0$, is a constant.

(ia) From the above condition is equivalent to $\| \phi(0) \| \leq \mathbb{L} \| \phi \|_{\mathcal{B}}$, for all $\phi \in \mathcal{B}$.

(ii) There exists a continuous function $c_1(t) \geq 0$ and a locally bounded function $c_2(t) \geq 0$ in $t \geq 0$ such that

$$\| x_t \|_{\mathcal{B}} \leq c_1(t) \sup_{s \in [0,t]} |x(s)| + c_2(t) \| x_0 \|_{\mathcal{B}}, \text{for } t \in [0, T] \text{ and } x \text{ as in (i).}$$

(iii) The space B is complete.

Designate $c_1^* = \sup\{c_1(t): t \in J\}$ and $c_2^* = \sup\{c_2(t): t \in J\}$.
Set

$$\mathbb{R}(\rho^-) = \{\rho(s, \phi): (s, \phi) \in J \times \mathcal{B}, \rho(s, \phi) \leq 0\}.$$

We generally assume that $\rho: J \times \mathcal{B} \to (-\infty, 0]$ is continuous, we make the subsequent assumption:

(H0) The function $t \to \phi$ is continuous from $\mathbb{R}(\rho^-)$ into $\mathcal{B}$ and we can find a continuous and bounded function $L^\phi: \mathbb{R}(\rho^-) \to (0, \infty)$ in a way that

$$\| \phi_t \|_{\mathcal{B}} \leq L^\phi(t) \| \phi \|_{\mathcal{B}} \quad \text{forevery } t \in \mathbb{R}(\rho^-).$$

Lemma 2.1 *[4] If $x: (-\infty, T] \to X$ is a function to ensure that $x_0 = \phi$, then*

$$\| x_s \|_B \leq (c_2^* + L^\phi) \| \phi \|_{\mathcal{B}} + c_1^* \sup_{\theta \in [0,s]} |x(\theta)|, \quad s \in \mathbb{R}(\rho^-) \cup J,$$

where $L^\phi = \sup_{t \in R(\rho^-)} L^\phi(t)$.

Let I be the identity operator on X. If A is a linear operator on X, then $R(\lambda, A) = (\lambda I - A)^{-1}$ means the resolvent operator of A. Next, we utilize the note for $\eta > 0$,

$$k_\eta(t) = \frac{t^{\eta-1}}{\Gamma(\eta)}, \quad t > 0, \tag{2.1}$$

where $\Gamma(\eta)$ is the Gamma function. If $\eta = 0$, we fixed $k_0(t) = \delta(t)$, the delta distribution.

Next, we recall some basic definitions and concepts of fractional integral and derivative of order $\alpha \in (1,2]$ from [21, Definitions 2.1 - 2.7, Remark 2.1].

Before we define the mild solution for the system (1.1) - (1.2), first we consider the following linear problem

$$
\begin{gathered}
{}^{C}D_t^{\alpha}[x(t)+U(t)] = A[x(t)+U(t)] + V(t), \quad t \in [0,T], \\
x(t) = \phi(t), \quad t \in B. \quad x'(0) = y_0 \in X.
\end{gathered}
$$

Assume that the Laplace transform of $x(t), U(t)$ and $V(t)$, with respect to t exists. Taking the Laplace transform to (2.2) - (2.3), by (2.3) of [21], we receive

$$\lambda^{\alpha}[\hat{x}(\lambda) + \widehat{U}(\lambda)] - \lambda^{\alpha-1}[x(0) + U(0)] - \lambda^{\alpha-2}[y_0 + \eta] = A[\hat{x}(\lambda) + \widehat{U}(\lambda)] + \hat{V}(\lambda),$$

where $\hat{x}(\lambda), \widehat{U}(\lambda)$ and $\hat{V}(\lambda)$ denote the Laplace transform of $x(t), U(t), V(t)$ and $\frac{d}{dt}U(t)|_{t=0} = \eta$, where η is independent of x. Then

$$\hat{x}(\lambda) + \widehat{U}(\lambda) = \lambda^{\alpha-1}R(\lambda^{\alpha}, A)[\phi + U(0)] + \lambda^{\alpha-2}R(\lambda^{\alpha}, A)[y_0 + \eta] + R(\lambda^{\alpha}, A)\hat{V}(\lambda).$$

By (2.5) - (2.7) of [21] and the properties of Laplace transforms,

$$x(t) = C_{\alpha}(t)[\phi + U(0)] + S_{\alpha}(t)[y_0 + \eta] - U(t) + \int_0^t P_{\alpha}(t-s)V(s)ds. \tag{2.4}$$

Let

$$S_{V,x} = \{v \in L^1(J,X) \colon v(t) \in V(t,x), \text{ a.e. } t \in J\}$$

is nonempty.

Let $\Omega = \{x \colon (-\infty, T] \to X$ such that $x|_{(-\infty,0]} \in \mathcal{B}, x|_J \in C(J,X)\}$.
Based on the above results, we define the mild solution for the given system (1.1) - (1.2).

Definition 2.1 *We say that a continuous function $x \in \Omega$ is a mild solution of problem (1.1) - (1.2) if $x(t) = \phi(t)$ for all $t \leq 0$, the constraint of $x(\cdot)$ to the interval $[0,T]$ is continuous and there exists $v(\cdot) \in L^1(J,X)$, such that $v(t) \in V(t, x_{\rho(t,x_t)})$ a.e. $t \in [0,T]$ and x fulfills the consecutive integral equation*

$$x(t) = C_{\alpha}(t)[\phi(0) + U(0,\phi(0))] + S_{\alpha}(t)[y_0 + \eta] - U(t,x_t) + \int_0^t P_{\alpha}(t-s)v(s)ds, t \in J. \tag{2.5}$$

3 Main Results

We show below the existence results for the systems (1.1) - (1.2) and (1.3) - (1.4) under Dhage's fixed point theorem.

To establish the existence result for the system (1.1) - (1.2), we list the subsequent conditions:

(H1) $A \colon D(A) \subset X \to X$ is the infinitesimal generator of a uniformly continuous cosine family $\{C_\alpha(t)\}_{t\geq 0}$. Let

$$M_c = \sup\{\| C_\alpha(t) \|; t \geq 0\} \quad \text{and} \quad M_s = \sup\{\| S_\alpha(t) \|; t \geq 0\}.$$

(H2) (i) The multi-valued map $V: J \times \mathcal{B} \to \mathcal{P}_{cv}(X)$ is an L^1-Caratheodory function and there exists a function $\mu \in L^1(J, \mathbb{R}^+)$ and a continuous non-decreasing function $\psi: \mathbb{R}^+ \to (0, \infty)$ in a way that

$$V(t,u) \leq \mu(t)\psi(\| u \|_{\mathcal{B}}), \quad \text{for every} \quad (t,u) \in J \times \mathcal{B}.$$

(ii) We can find $L_{f_1} \in L^1(J, \mathbb{R}^+)$ to ensure that

$$\| V(t,u) - V(t,v) \| \leq L_{f_1} \| u - v \|_{\mathcal{B}} \quad \text{forall} \quad u, v \in \mathcal{B}.$$

(H3) (i) The function $U: J \times \mathcal{B} \to X$ is continuous on J and there exist constants $L_U > 0$ and $\tilde{L}_U > 0$ such that

$$\| U(t,u) \| \leq L_U \| u \|_{\mathcal{B}} + \tilde{L}_U, \quad \text{foreach} \quad u \in \mathcal{B}.$$

(ii) There exist a function $L_U^* \in L^1(J, \mathbb{R}^+)$ in a way that

$$\| U(t,u) - U(t,v) \| \leq L_U^* \| u - v \|_{\mathcal{B}}, \quad t \in J, \ u, v \in \mathcal{B}.$$

Theorem 3.1 *Assume that the hypotheses (H0) and (H1) - (H3) holds. Then the problem (1.1) - (1.2) has at least one mild solution such that*

$$\Lambda = L_U^* c_1^* < 1 \quad \text{and} \quad \int_0^T \mu(s)ds < \int_{\omega_1}^{\infty} \frac{du}{\psi(u)}, \tag{3.1}$$

where $\omega_1 = c_n + \frac{c_1^*}{1 - L_U c_1^*}\{M_c(L_U \mathbb{L} \| \phi \|_{\mathcal{B}} + \tilde{L}_U) + M_s \| y_0 + \eta \| + L_U c_n^* + \tilde{L}_U\}$.

Proof. Now, we define the the multi-valued operator $\Upsilon: \Omega \to P(\Omega)$ described by $\Upsilon(e) = \{e \in \Omega\}$ with

$$e(t) = \begin{cases} \phi(t), & t \in (-\infty, 0] \\ C_\alpha(t)[\phi(0) + U(0, \phi(0))] + S_\alpha(t)[x_0 + \eta] - U(t, x_t) \\ + \int_0^t P_\alpha(t-s)v(s)ds, & t \in J, \end{cases} \tag{3.2}$$

where $v \in S_{V, x_{\rho(s,x_s)}}$. For $\phi \in \mathcal{B}$, we express the function $y(\cdot): (-\infty, T] \to X$ by

$$y(t) = \begin{cases} \phi(t), & t \leq 0 \\ C_\alpha(t)\phi(0), & t \in J. \end{cases}$$

and then $y_0 = \phi$. For every function $z \in C(J, X)$ with $\overline{z}_0 = 0$, we denote by $\overline{z}$ by the function described by

$$\bar{z}(t) = \begin{cases} 0, & t \leq 0 \\ z(t), & t \in J. \end{cases}$$

If $x(\cdot)$ fulfills (2.5), we can able to decompose $x(t) = y(t) + z(t), t \in J$, with infer that $x_t = y_t + z_t$, for each $t \in J$ and the function $z(\cdot)$ fulfills

$$z(t) = C_\alpha(t)U(0, \phi(0)) + S_\alpha(t)[y_0 + \eta] - U(t, z_t + y_t) + \int_0^t P_\alpha(t-s)v(s)ds, \quad t \in J. \quad (3.3)$$

where $v \in S_{V, z_{\rho(s,z_s+y_s)}+y_{\rho(s,z_s+y_s)}}$.

Let $\mathcal{Z}_0 = \{z \in \Omega: z_0 = 0\}$. For any $z \in \mathcal{Z}_0$, we receive

$$\| z \|_{\mathcal{Z}_0} = \sup_{t\in J} \| z(t) \| + \| z_0 \|_{\mathcal{B}} = \sup_{t\in J} \| z(t) \|.$$

Therefore $(\mathcal{Z}_0, \|\cdot\|_{\mathcal{Z}_0})$ is a Banach space. Now, we designate the operator $\Phi: \mathcal{Z}_0 \to \mathcal{P}(\mathcal{Z}_0)$ by $\Phi(z) = \{h \in \mathcal{Z}_0\}$ with

$$h(t) = C_\alpha(t)U(0, \phi(0)) + S_\alpha(t)[y_0 + \eta] - U(t, z_t + y_t) + \int_0^t P_\alpha(t-s)v(s)ds, \quad t \in J,$$

where $v \in S_{V, z_{\rho(s,z_s+y_s)}+y_{\rho(s,z_s+y_s)}}$.

From this, we observe that the operator Υ having a fixed point is equivalent to Φ having one, so it turns to prove that Φ has a fixed point.

Now, we are in a position to utilize the Dhage's fixed point theorem [13]. To apply this, first, we split the multi-valued operator Φ as

$$\Phi_1(z) = \{h \in \mathcal{Z}_0: h(t) = C_\alpha(t)U(0, \phi(0)) + S_\alpha(t)[y_0 + \eta] - U(t, z_t + y_t), \ t \in J\},$$
$$\Phi_2(z) = \left\{h \in \mathcal{Z}_0: h(t) = \int_0^t P_\alpha(t-s)v(s)ds, \ v(s) \in S_{V, z_{\rho(s,z_s+y_s)}+y_{\rho(s,z_s+y_s)}}, t \in J\right\}.$$

Now, our aim is to show that the multi-valued operators Φ_1 and Φ_2 satisfy all the conditions of Dhage's fixed point theorem [13]. For better readability, we break the proof into sequence of steps.

Step 1. Φ_1 is a contraction.

Let $z, z^* \in \mathcal{Z}_0$ and $h \in \Phi_1(z)$

$$\begin{aligned} \| \Phi_1(z) - \Phi_1(z^*) \| &\leq \| U(t, z_t + y_t) - U(t, z_t^* + y_t) \|_{\mathcal{B}} \\ &\leq L_U^* \| z_t - z_t^* \|_{\mathcal{B}} \\ &\leq L_U^* c_1(t) \sup_{t\in J} |z(t) - z^*(t)| + c_2(t) \| z_0 - z_0^* \| \\ &\leq L_U^* c_1^* \| z - z^* \|_{\mathcal{B}}, \end{aligned}$$

since

$$\begin{aligned}\| z_t + y_t \|_{\mathcal{B}} &\leq \| z_t \|_{\mathcal{B}} + \| y_t \|_{\mathcal{B}} \\ &\leq c_1^* \sup_{t\in J}|z(t)| + c_2(t) \| z_0 \|_{\mathcal{B}} + c_1^* \sup_{t\in J}|y(s)| + \| y_0 \|_{\mathcal{B}} \\ &\leq c_1^* \sup_{t\in J}|z(t)| + c_1^* |C_\alpha(s)\phi(0)|_{\mathcal{B}} + c_2^* \| C_\alpha(0)\phi \|_{\mathcal{B}} \\ &\leq c_1^* \sup_{t\in J}|z(t)| + c_1^* M_c \mathbb{L} \| \phi \|_{\mathcal{B}} + c_2^* \| \phi \|_{\mathcal{B}} \\ &\leq c_1^* r + [c_1^* M_c \mathbb{L} + c_2^*] \| \phi \|_{\mathcal{B}} \\ &\leq c_1^* r + c_n^*, \text{ where } c_n^* = [c_1^* M_c \mathbb{L} + c_2^*] \| \phi \|_{\mathcal{B}}.\end{aligned}$$

Then

$$\| \Phi_1(z) - \Phi_1(z^*) \| \leq \Lambda \| z - z^* \|.$$

From (3.1), we see that Φ_1 is a contraction.

Step 2. Φ_2 has compact, convex valued and it is completely continuous. This will show in several steps.

Claim (i). Φ_2 is convex for each $z \in \mathcal{Z}_0$.

Indeed, if $h_1, h_2 \in \Phi_2$, then there exist $v_1, v_2 \in S_{V, z_{\rho(t,z_t+y_t)}+y_{\rho(t,z_t+y_t)}}$, such that for $t \in J$, we receive

$$h_i = \int_0^t P_\alpha(t-s)v_i(s)ds, \quad \text{for } i = 1,2, \quad t \in J.$$

Let $d \in [0,1]$. Then for each $t \in J$, we get

$$[dh_1 + (1-d)h_2](t) = \int_0^t P_\alpha(t-s)[dv_1(s) + (1-d)v_2(s)]ds.$$

Actually $S_{V, z_{\rho(t,z_t+y_t)}+y_{\rho(t,z_t+y_t)}}$ is convex (because F has convex values), we have $dh_1 + (1-d)h_2 \in \Phi_2$.

Claim (ii). Φ_2 maps bounded sets into bounded sets in $\mathcal{Z}_0$.

In fact, it is sufficient to demonstrate that for any $r > 0$, there exists a positive constant l in ways that for every $z \in B_r = \{z \in \mathcal{Z}_0 : \| z \|_{\mathcal{Z}_0} \leq r\}$, we sustain $\| \Phi_2(z) \|_{\mathcal{Z}_0} \leq l$. Then for every $h \in \Phi_2(z)$, there exists $v \in S_{V, z_{\rho(t,z_t+y_t)}+y_{\rho(t,z_t+y_t)}}$ such that

$$\begin{aligned}\| h(t) \| &\leq \| \int_0^t P_\alpha(t-s)v(s)ds \| \\ &\leq \int_0^t \| P_\alpha(t-s)v(s) \| ds \\ &\leq M \int_0^t \mu(s)\psi\big(\| z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)} \|_{\mathcal{B}}\big)ds \\ &\leq M\psi(c_1^* r + c_n) \int_0^t \mu(s)ds \\ &\leq M\psi(c_1^* r + c_n) \| \mu \|_{L^1} \\ &\leq l,\end{aligned}$$

since

$$
\begin{aligned}
&\| z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)} \|_{\mathcal{B}} \\
&\quad \leq \| z_{\rho(s,z_s+y_s)} \|_{\mathcal{B}} + \| y_{\rho(s,z_s+y_s)} \|_{\mathcal{B}} \\
&\quad \leq (c_2^* + L^{\phi}) \| z_0 \| + c_1^* \sup_{s\in t}|z(s)| + c_1^* \sup_{s\in t}|y(s)| + (c_2^* + L^{\phi}) \| y_0 \|_{\mathcal{B}} \\
&\quad \leq c_1^* \sup_{s\in t}|z(s)| + c_1^*|C_\alpha(s)\phi(0)| + (c_2^* + L^{\phi}) \| C_\alpha(0)\phi \|_{\mathcal{B}} \\
&\quad \leq c_1^* \sup_{s\in t}|z(s)| + c_1^* M_c \mathbb{L} \| \phi \|_{\mathcal{B}} + (c_2^* + L^{\phi}) \| \phi \|_{\mathcal{B}} \\
&\quad \leq c_1^* r + [c_1^* M_c \mathbb{L} + c_2^* + L^{\phi}] \| \phi \|_{\mathcal{B}} \\
&\quad \leq c_1^* r + c_n, \quad \text{where} \quad c_n = [c_1^* M_c \mathbb{L} + c_2^* + L^{\phi}] \| \phi \|_{\mathcal{B}}.
\end{aligned}
$$

Thus $\Phi_2(B_r)$ is bounded.

Claim (iii). Φ_2 maps bounded sets into equi-continuous sets of $\mathcal{Z}_0$.

Let $h \in \Phi_2(z)$ for $z \in \mathcal{Z}_0$ and let $\tau_1, \tau_2 \in [0,T]$, with $\tau_1 < \tau_2$, we have

$$
\begin{aligned}
\| h(\tau_2) - h(\tau_1) \| &\leq \| \int_0^{\tau_2} P_\alpha(\tau_2 - s)v(s)ds - \int_0^{\tau_1} P_\alpha(\tau_1 - s)v(s)ds \| \\
&\leq \| \int_0^{\tau_1-\epsilon} [P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s)]v(s)ds \\
&\quad + \int_{\tau_1-\epsilon}^{\tau_1} [P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s)]v(s)ds \\
&\quad + \int_{\tau_1}^{\tau_2} P_\alpha(\tau_2 - s)v(s)ds \| \\
&\leq \| \int_0^{\tau_1-\epsilon} [P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s)]v(s)ds \| \\
&\quad + \| \int_{\tau_1-\epsilon}^{\tau_1} [P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s)]v(s)ds \| \\
&\quad + \| \int_{\tau_1}^{\tau_2} P_\alpha(\tau_2 - s)v(s)ds \| \\
&\leq \psi(c_1^* r + c_n)\{\int_0^{\tau_1-\epsilon} \| P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s) \| \mu(s)ds \\
&\quad + \int_{\tau_1-\epsilon}^{\tau_1} \| P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s) \| \mu(s)ds \\
&\quad + M \int_{\tau_1}^{\tau_2} \mu(s)ds\}.
\end{aligned}
$$

Clearly the right-hand side of the above inequality tends to zero as $\tau_2 \to \tau_1$.

Then, $\Phi_2(B_r)$ is equi-continuous.

Claim (iv). $\Phi_2(B_r)$ is relatively compact for every $t \in J$, we sustain

$$\Phi_2(B_r)(t) = \{h(t): h \in \Phi_2(B_r)\}.$$

Allow $0 \leq t \leq T$ be fixed and let ϵ be a real number fulfilling $0 < \epsilon < t$. For $\delta > 0$, we specify

$$h_{\epsilon,\delta}(t) = \int_0^{t-\epsilon} P_\alpha(t - s)v(s)ds,$$

where $v(s) \in S_{V,z_{\rho(s,z_s+y_s)}+y_{\rho(s,z_s+y_s)}}$. Since $C_\alpha(t)$ is a compact operator,

$$h_{\epsilon,\delta} = \{h_{\epsilon,\delta}(t): h \in \Phi_2(B_r)\}$$

is relatively compact. Furthermore,

$$\begin{aligned}\| h(t) - h_{\epsilon,\delta}(t) \| &\leq \| \int_0^t P_\alpha(t-s)v(s)ds - \int_0^{t-\epsilon} P_\alpha(t-s)v(s)ds \| \\ &\leq \| \int_{t-\epsilon}^t P_\alpha(t-s)v(s)ds \| \\ &\leq \int_{t-\epsilon}^t \| P_\alpha(t-s) \| \| v(s) \| ds \\ &\leq M \int_{t-\epsilon}^t \mu(s)\psi\big(\| z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)} \|_B\big)ds \\ &\leq M\psi(c_1^* r + c_n) \int_{t-\epsilon}^t \mu(s)ds.\end{aligned}$$

Therefore, $(\Phi_2(B_r))(t)$ is relatively compact.

As a consequence of claim (ii) - (iv) together with the Arzela-Ascoli's theorem we can conclude that Φ_2 is completely continuous.

Claim (v). Φ_2 has a closed graph.

Let $z_n \to z_*, h_n \in \Phi_2(z_n)$ and $h_n \to h_*$. We will prove that $h_* \in \Phi_2(z_*)$. Indeed, $h_n \in \Phi_2(z_n)$ means that there exists $v_n \in S_{V,z_{\rho(n,z_n+y_n)}+y_{\rho(n,z_n+y_n)}}$ such that

$$h_n(t) = \int_0^t P_\alpha(t-s)v_n(s)ds, \quad t \in J.$$

We must prove that there exists $v_* \in S_{V,z_{\rho(*,z_*+y_*)}+y_{\rho(*,z_*+y_*)}}$ such that

$$h_*(t) = \int_0^t P_\alpha(t-s)v_*(s)ds, t \in J.$$

Consider the linear and continuous operator $\mathcal{Y}: L^1(J,X) \to C(J,X)$, defined by

$$\mathcal{Y}(v)(t) = \int_0^t P_\alpha(t-s)v(s)ds.$$

From[24], it follows that $\mathcal{Y} \circ S_F$ is a closed graph operator and from the definition of $h_n(t) \in \mathcal{Y}S_{V,z_{\rho(n,z_n+y_n)}+y_{\rho(n,z_n+y_n)}}$.

As $z_n \to z_*$ and $h_n \to h_*$, there is a $v_* \in S_{V,z_{\rho(*,z_*+y_*)}+y_{\rho(*,z_*+y_*)}}$ such that

$$h_*(t) = \int_0^t P_\alpha(t-s)v(s)ds.$$

Therefore, the multi-valued operator Φ_2 is a completely continuous multivalued map, upper semi-continuous with convex, closed and compact values.

Claim (vi). A priori bounds.

$$\Gamma = \{z \in \mathcal{Z}_0 : z \in \lambda\Phi_1(z) + \lambda\Phi_2(z), \text{ forsome } 0 < \lambda < 1\} \text{ is bounded.}$$

Let $z \in \Gamma$ be any element, then there exists $v \in S_{V, z_{\rho(s,z_s+y_s)}+y_{\rho(s,z_s+y_s)}}$, such that

$$\begin{aligned}\| z(t) \| &\le \| C_\alpha(t)U(0,\phi(0)) + S_\alpha(t)[y_0+\eta] - U(t, z_t + y_t) \\ &\quad + \int_0^t P_\alpha(t-s)V(s, z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)})ds \| \\ &\le M_c(L_U \| \phi(0) \|_{\mathcal{B}} + \tilde{L}_U) + M_s \| y_0+\eta \| + L_U \| z_t + y_t \|_{\mathcal{B}} + \tilde{L}_U \\ &\quad + M\int_0^t \mu(s)\psi\big(\| z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)} \|_{\mathcal{B}}\big)ds \\ &\le M_c(L_U\mathbb{L} \| \phi \|_{\mathcal{B}} + \tilde{L}_U) + M_s \| y_0+\eta \| + L_U(c_1^*\sup_{t\in J}|z(t)| + c_n^*) + \tilde{L}_U \\ &\quad + M\int_0^t \mu(s)\psi(c_1^*\sup_{s\in[0,t]}|z(s)| + c_n)ds \\ &\le \frac{1}{1-L_U c_1^*}\{M_c(L_U\mathbb{L} \| \phi \|_{\mathcal{B}} + \tilde{L}_U) + M_s \| y_0+\eta \| + L_U c_n^* + \tilde{L}_U \\ &\quad + M\int_0^t \mu(s)\psi(c_1^* \| z(s) \| + c_n)ds\}.\end{aligned}$$

Then

$$\begin{aligned}c_n + c_1^* \| z(t) \| &\le c_n + \frac{c_1^*}{1-L_U c_1^*}\{M_c(L_U\mathbb{L} \| \phi \|_{\mathcal{B}} + \tilde{L}_U) + M_s \| y_0+\eta \| + L_U c_n^* + \tilde{L}_U\} \\ &\quad + \frac{Mc_1^*}{1-L_U c_1^*}\int_0^t \mu(s)\psi(c_n + c_1^* \| z(s) \|)ds \\ &\le \omega_1 + \omega_2\int_0^t \mu(s)\psi(c_n + c_1^* \| z(s) \|)ds,\end{aligned}$$

where

$$\omega_1 = c_n + \frac{c_1^*}{1-L_U c_1^*}\{M_c(L_U\mathbb{L} \| \phi \|_{\mathcal{B}} + \tilde{L}_U) + M_s \| y_0+\eta \| + L_U c_n^* + \tilde{L}_U\}$$

and

$$\omega_2 = \frac{Mc_1^*}{1-L_U c_1^*}.$$

Denote

$$m(t) = \sup\{c_n + c_1^* \| z(s) \| : 0 \le s \le t\}, \ t \in J.$$

From the above mentioned inequality, we receive

$$m(t) \le \omega_1 + \omega_2\int_0^t \mu(s)\psi(m(s))ds.$$

Let us take the right hand side of the above inequality as $\nu(t)$. Thus, we get

$$m(t) \le \nu(t), \text{ forevery } t \in J,$$

with

$$\nu(0) = \omega_1$$

and

$$\nu'(t) = \omega_2\mu(t)\psi(m(t)), \text{ a.e. } t \in J.$$

Utilizing the non-decreasing character of ψ, we obtain

$$\nu'(t) \le \omega_2 \mu(t)\psi(\nu(t)), \text{ a.e. } t \in J.$$

Integrating from 0 to t we get

$$\int_0^t \frac{\nu'(s)}{\psi(\nu(s))} ds \le \omega_2 \int_0^t \mu(s)ds.$$

By change the variable, we get

$$\int_{\nu(0)}^{\nu(t)} \frac{du}{\psi(u)} \le \omega_2 \int_0^t \mu(s)ds.$$

In view of (3.1), this ensures that for every $t \in J$, we have

$$\int_{\nu(0)}^{\nu(t)} \frac{du}{\psi(u)} \le \omega_2 \int_0^t \mu(s)ds \le \omega_2 \int_0^T \mu(s)ds < \int_{\omega_1}^{\infty} \frac{du}{\psi(u)}.$$

Therefore, for every $t \in J$, there exists a constant Λ_1 such that $v(t) \le \Lambda_1$ and hence $m(t) \le \Lambda_1$. Since $\| z \|_{\mathcal{Z}_0} \le m(t)$, we have $\| z \|_{\mathcal{Z}_0} \le \Lambda_1$.

This shows that the set Γ is bounded. As a consequence of Dhage's fixed point theorem [13], we realize that $\Phi_1 + \Phi_2$ has a fixed point z defined on the interval $(-\infty, T]$ which is the mild solution of the system (1.1) - (1.2). The proof is now completed.

Our next existence results for the problem (1.3) - (1.4) is based on Dhage's fixed point theorem. Before we present and prove the existence result for the problem, first we define the mild solution of (1.3) - (1.4).

Definition 3.1 *We say that a continuous function $x \in \Omega$ is a mild solution of problem (1.3) - (1.4) if $x(0) = \phi \in \mathcal{B}, x'(0) = y_0 \in X$, we have*

$$x(t) = C_\alpha(t)[\phi(0) + U(0, \phi(0), 0)] + S_\alpha(t)[y_0 + \eta] - U(t, x_t, \int_0^t k_1(t, s, x_s)ds)$$
$$+ \int_0^t P_\alpha(t-s)V(s, x_{\rho(s,x_s)}, \int_0^s k_2(s, \tau, x_{\rho(\tau,x_\tau)})d\tau)ds, \quad t \in J. \tag{3.4}$$

Next, to prove the existence result for the problem (1.3) - (1.4), we list the following additional hypotheses:

(H2*) The multivalued map $V: J \times \mathcal{B} \times X \to \mathcal{P}_{cl,cv,bd}(X)$ is an L^1-Caratheodory function.

(H3*) $t \in J$, the function $V(t, \cdot, \cdot): \mathcal{B} \times X \to \mathcal{P}_{cl,cv,bd}(X)$ is upper semi continuous and for each $(u, v) \in \mathcal{B} \times X$, the function $F(\cdot, u, v): J \to X$ is strongly measurable. Also, for each fixed $u \in \mathcal{B}$ and the set

$$S_{V,x^*} = \{v^* \in L^1(J, X): v^*(t) \in V(t, x_{\rho(t,x_t)}, \int_0^t k_2(t, s, x_{\rho(s,x_s)})ds) \text{ fora.e. } t \in J\}$$

is nonempty.

(H4*) (i) There exist a continuous function $\mu_1 \in L^1(J, \mathbb{R}^+)$ and a continuous non decreasing function $\psi_1: \mathbb{R}^+ \to (0, \infty)$ in a way that

$$\| V(t,u,v) \|_X \leq \mu_1(t)\psi_1(\| u \|_{\mathcal{B}} + \| v \|_X), \text{ fora.e. } t \in J,\ u \in \mathcal{B}, v \in X,$$

(ii) We can find $L_f \in L^1(J, \mathbb{R}^+)$ to ensure that

$$\| V(t,u_1,v_1) - V(t,u_2,v_2) \|_X \leq L_f[\| u_1 - v_1 \|_{\mathcal{B}} + \| u_2 - v_2 \|_X],$$
$$\text{for a.e. } t \in J,\ u_1, v_1 \in \mathcal{B}, u_2, v_2 \in X.$$

(iii) There is a function $m \in L^1(J, \mathbb{R}^+)$ and a non decreasing function $\Omega_1: \mathbb{R}^+ \to (0, \infty)$ to ensure that

$$\| k_2(t,s,u) \|_X \leq m(s)\Omega_1(\| u \|_{\mathcal{B}}), \text{ forevery } (t,s,u) \in J \times J \times \mathcal{B}.$$

(iv) There is a constant $C_1 > 0$, in a way that

$$\| \int_0^t [k_2(t,s,u_1) - k_2(t,s,u_2)]ds \|_X \leq C_1 \| u_1 - u_2 \|_{\mathcal{B}}, \text{for } (t,s) \in J \times J,\ u_1, u_2 \in \mathcal{B}.$$

(H5*) (i) The function $U: J \times \mathcal{B} \times X \to X$ is continuous on J and there exist a positive constant $M_U > 0$ such that for each $u_1, v_1 \in \mathcal{B},\ u_2, v_2 \in X$.

$$\| U(t,u_1,v_1) - U(t,u_2,v_2) \| \leq M_U(\| u_1 - v_1 \|_{\mathcal{B}} + \| u_2 - v_2 \|_X).$$

(ii) There exist positive constants $L_{U_1} > 0$ and $\tilde{L}_{U_1} > 0$, such that

$$\| U(t,u,v) \| \leq L_{U_1}(\| u \|_{\mathcal{B}} + \| v \|_X) + \tilde{L}_{U_1}, \quad t \in J,\ u \in \mathcal{B},\ v \in X.$$

(H6*) The function $k_1: J \times J \times X \to X$ are continuous maps and there exists a positive constant $L_{k_1} > 0$ such that

$$\| \int_0^t [k_1(t,s,z_1) - k_1(t,s,z_2)]ds \|_X \leq L_{k_1} \| z_1 - z_2 \|_{\mathcal{B}}, \text{ foreach } z_1, z_2 \in \mathcal{B}$$

and

$$\| \int_0^t k_1(t,s,z)ds \|_X \leq L_{k_1}(\| z \|_{\mathcal{B}} + 1), \quad \text{for } z \in \mathcal{B}.$$

Theorem 3.2 *Assume that the hypotheses (H0), (H1) and (H2*) - (H6*) holds. Then the problem (1.3) - (1.4) has at least one mild solution such that*

$$\Lambda^* = M_U c_1^*[1 + (L_{k_1} + 1)] < 1 \quad \text{and} \int_0^T \nu(s)ds < \int_{\omega_1^*}^{\infty} \frac{ds}{\psi_1(s) + \Omega_1(s)},$$

where

$$\omega_1^* = c_n + \frac{c_1^*}{1 - L_{U_1} c_1^*(1 + L_{k_1})}\{M_c(L_{U_1}\mathbb{L} \| \phi \|_{\mathcal{B}} + \tilde{L}_{U_1}) + M_s \| y_0 + \eta \| + L_{U_1} c_n^*(1 + L_{k_1})\}.$$

(3.5)

Proof. Consider the multivalued operator $\overline{\Upsilon}:\Omega \to P(\Omega)$ defined by $\overline{\Upsilon}(h) = \{h \in \Omega\}$ with

$$h(t) = \begin{cases} \phi(t), & t \in (-\infty, 0] \\ C_\alpha(t)[\phi(0) + U(0,\phi(0),0)] + S_\alpha(t)[y_0 + \eta] & \\ -U(t, x_t, \int_0^t k_1(t,s,x_s)ds) + \int_0^t P_\alpha(t-s)v^*(s)ds, & t \in J, \end{cases}$$

where $v^*(s) \in S_{V,x^*}$.

Next, we split the multi-valued operator Φ^* as

$$\Phi_1^*(z) = \{h \in \mathcal{Z}_0 : h(t) = C_\alpha(t)U(0,\phi(0),0) + S_\alpha(t)[y_0 + \eta] - U(t, x_t, \int_0^t k_1(t,s,x_s)ds),\ \ t \in J\}.$$
$$\Phi_2^*(z) = \{h \in \mathcal{Z}_0 : h(t) = \int_0^t P_\alpha(t-s)v^*(s)ds,\ \ v^*(s) \in S_{F,x^*},\ t \in J\}.$$

The proof of this theorem is very similar to Theorem 3.1. With necessary modifications, we can prove the steps 1 and 2 (Claims (i)-(v)) clearly, so we omit these steps. Now, we prove a priori bounds only.

Claim (vi). A priori bounds.

$$\Gamma = \{z \in \mathcal{Z}_0 : z \in \lambda\Phi_1^*(z) + \lambda\Phi_2^*(z),\ \text{forsome}\ \ 0 < \lambda < 1\}\ \ \text{is bounded.}$$

Let $z \in \Gamma$ be any element, then there exists $v^* \in S_{V,x^*}$, such that

$$\begin{aligned} \| z(t) \| \le \| & C_\alpha(t)U(0,\phi(0),0) + S_\alpha(t)[y_0+\eta] - U(t, z_t + y_t, \int_0^t k_1(t,s,z_s+y_s)ds) \\ & + \int_0^t P_\alpha(t-s)V(s, z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)}, \int_0^s k_2(s,\tau, z_{\rho(\tau,z_\tau+y_\tau)} + y_{\rho(\tau,z_\tau+y_\tau)})d\tau)ds \| \end{aligned}$$

$$\begin{aligned} \le & \| C_\alpha(t) \| \| U(0,\phi(0),0) \| + \| S_\alpha(t)[y_0+\eta] \| + \| U(t, z_t + y_t, \int_0^t k_1(t,s,z_s+y_s)ds) \| \\ & + \| \int_0^t P_\alpha(t-s)V\left(s, z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)}, \int_0^s k_2\big(s,\tau, z_{\rho(\tau,z_\tau+y_\tau)} + y_{\rho(\tau,z_\tau+y_\tau)}\big)d\tau\right)ds \| \end{aligned}$$

$$\begin{aligned} \le & M_c(L_{U_1}\mathbb{L} \| \phi \|_{\mathcal{B}} + \tilde{L}_{U_1}) + M_s \| y_0 + \eta \| + L_{U_1}\left[\| z_t + y_t \|_{\mathcal{B}} + \| \int_0^t k_1(t,s,z_s+y_s)ds \|_X \right] + \tilde{L}_{U_1} \\ & + M\int_0^t \mu_1(s)\psi_1[\| z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)} \|_{\mathcal{B}} + \int_0^s \| k_2(s,\tau, z_{\rho(\tau,z_\tau+y_\tau)} + y_{\rho(\tau,z_\tau+y_\tau)}) \|_H \, d\tau]ds \end{aligned}$$

$$\leq M_c\big(L_{U_1}\mathbb{L} \parallel \phi \parallel_{\mathcal{B}} + \tilde{L}_{U_1}\big) + M_s \parallel y_0 + \eta \parallel + L_{U_1} c_1^* \sup_{t\in J} \parallel z(t) \parallel + L_{U_1} c_n^* + L_{U_1} L_{k_1} c_1^* \sup_{t\in J} \parallel z(t) \parallel + L_{U_1} L_{k_1} c_n^* + M \int_0^t \mu_1(s)\psi_1[c_1^* \parallel z(s) \parallel + c_n + \int_0^s m(\tau)\Omega_1(c_1^* \parallel z(\tau) \parallel + c_n)]]ds.$$

Then $c_n + c_1^* \parallel z(t) \parallel$

$$\leq c_n + \frac{c_1^*}{1 - L_{U_1} c_1^*(1 + L_{k_1})}\{M_c(L_{U_1}\mathbb{L} \parallel \phi \parallel_{\mathcal{B}} + \tilde{L}_{U_1}) + M_s \parallel y_0 + \eta \parallel + L_{U_1} c_n^*(1 + L_{k_1})\} + \frac{c_1^* M}{1 - L_{U_1} c_1^*(1 + L_{k_1})}\int_0^t \mu_1(s)\psi_1\{c_1^* \parallel z(s) \parallel + c_n + \int_0^s m(\tau)\Omega_1(c_1^* \parallel z(\tau) \parallel + c_n)d\tau\}ds,$$

where

$$\omega_1^* = c_n + \frac{c_1^*}{1 - L_{U_1} c_1^*(1 + L_{k_1})}\{M_c(L_{U_1}\mathbb{L} \parallel \phi \parallel_{\mathcal{B}} + \tilde{L}_{U_1}) + M_s \parallel y_0 + \eta \parallel + L_{U_1} c_n^*(1 + L_{k_1})\}$$

And

$$\omega_2^* = \frac{c_1^* M}{1 - L_{U_1} c_1^*(1 + L_{k_1})}.$$

Denote

$$\beta(t) = \sup\{c_n^* + D_1^* \parallel z(s) \parallel : 0 \leq s \leq t\}, \qquad t \in J.$$

From the above inequality, we receive

$$\beta(t) \leq \omega_1^* + \omega_2^* \int_0^t \mu_1(s)\psi_1\{\beta(s) + \int_0^s m(\tau)\Omega_1(\beta(\tau))d\tau\}ds.$$

Let us take the right hand side of the above inequality as $W(t)$. Thus, we get

$$\beta(t) \leq W(t), \text{ forevery } t \in J,$$

with

$$W(0) = \omega_1^*$$

And

$$W'(t) = \omega_2^* \mu_1(t)\psi_1\{W(t) + \int_0^t m(s)\Omega_1(W(s))ds\}, \text{ a.e. } t \in J.$$

Utilizing the non-decreasing character of ψ_1, we obtain

$$W'(t) \leq \omega_2^* \mu_1(t)\psi_1(W(t)) + m(t)\Omega_1(W(t)), \text{ a.e. } t \in J.$$

We characterize the function $\gamma(t) = \max\{\omega_2^* \mu_1(t), m(t)\}$, $t \in J$, we suggests that

$$\frac{W'(t)}{\psi_1(W(t)) + \Omega_1(W(t))} \leq \gamma(t).$$

Integrating from 0 to t we get

$$\int_0^t \frac{W'(s)ds}{\psi_1(W(s))+\Omega_1(W(s))} \leq \int_0^t \gamma(s)ds.$$

In view of (3.5), this implies that for each $t \in J$, we sustain

$$\int_{W(0)}^{W(t)} \frac{ds}{\psi_1(s)+\Omega_1(s)} \leq \int_0^t \gamma(s)ds \leq \int_0^T \gamma(s)ds < \int_{\omega_1^*}^{\infty} \frac{ds}{\psi_1(s)+\Omega_1(s)}.$$

Therefore, for each $t \in J$, there exists a constant Λ_1 in a way that $W(t) \leq \Lambda_2$ and hence $\beta(t) \leq \Lambda_2$. Since $\parallel z \parallel_{Z_0} \leq \beta(t)$, we have $\parallel z \parallel_{Z_0} \leq \Lambda_2$.

From this, we observe that the set Γ is bounded. As a consequence of Dhage's fixed point theorem[13], we realize that $\Phi_1^* + \Phi_2^*$ has a fixed point z defined on the interval $(-\infty, T]$ which is the mild solution of the problem (1.3) - (1.4).

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