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APPLICATION OF FUNCTION WITH NEW CONDITION IN PROBABILISTIC SPACE

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Abstract

In a wide range of mathematical computational modeling and engineering problems the existence of a solution to a theoretical problem is equivalent to the existence of fixed point of a map. Fixed point technique is very useful in science and engineering. In this paper we prove unique fixed point of a self-mapping in probabilistic metric space space by using contraction theory and using it in control theory as application.

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FIXED POINT OF A MAPPING [R P Pant 2002]:

Fixed point theorem, any of various theorems in mathematics dealing with the transformation of a set of points into the same set of points for which at least one point can be shown to remain fixed. For example, if every real number is squared, the numbers zero and one remain fixed; while a transformation in which each number is increased by one leaves no number fixed. In the first example, the transformation of squaring each number when applied to the open interval between numbers greater than and less than zero $(0,1)$ also has no fixed points. However, the situation changes for the closed interval $[0,1]$ according to the end points. A continuous transformation is one where adjacent points are transformed into other neighboring points. Brouwer's fixed point theorem states that a continuous transformation of a closed plate (including the boundary) onto itself leaves at least one point fixed. The theorem also applies to continuous transformations of points on a closed interval, a closed sphere, or abstract higher-dimensional sets analogous to a sphere. Fixed point theorems are very useful for determining whether an equation has a solution. For example, in differential equations, a transformation called a differential operator transforms a function into another. Finding a solution to a differential equation can then be interpreted as finding a function invariant by an associated transformation. Treating these functions as points and defining a set of functions analogous to the set of points consisting of the slab above, one can prove theorems for differential equations analogous to Brouwer's fixed point theorem. The best-known theorem of this type is the Leray-Schauder theorem, published in 1934 by the Frenchman Jean Leray and the Polish Julius Schauder. Whether or not this method yields a solution (ie whether a fixed point is found or not) depends on the exact nature of the differential operator and the set of functions of which the solution is sought.If we take a mapping H, here all the points of domain are being mapped to range set then set of all those points which are being mapped to itself are called fixed point of mapping H. There are very useful applications of fixed points in areas of engineering and physics. Solution of any differential equation can be assured.

FIXED POINT APPLICATION IN DIFFERENT DOMAINS [R P Pant 2002]:

The presence of a theoretical or practical solution is comparable to the existence of a fixed point for a suitable map or operator in a broad variety of mathematical, computing, economic, modelling, and engineering issues. Therefore, fixed points are crucial in many branches of mathematics, science, and engineering.The theory itself is a lovely synthesis of topology, geometry, and analysis (pure and applied). The theory of fixed points has emerged during the last 60 years or more as a highly potent and crucial tool in the study of nonlinear events. Fixed point methods in particular have been used in a wide range of disciplines, including biology, chemistry, physics, engineering, game theory, and economics. It is sometimes impossible to discover a precise answer, hence it is required to create suitable algorithms to approximatively produce the desired outcome. This has a close connection to control and optimisation issues that crop up in many sciences and engineering issues. Fixed point problems or optimisation may be used to develop a number of circumstances in the study of nonlinear equations, calculus of variations, partial differential equations, optimum control, and inverse issues.

By converting them to an analogous fixed-point issue, many nonlinear functional equation-based scientific and engineering difficulties may be resolved. In fact, a fixed-point equation $Hx = x$may be used to represent an operator equation $Tx = 0$, where H is a self-mapping with an appropriate domain. For example, split feasibility issues, variational inequality issues, nonlinear optimisation issues, equilibrium issues, complementarity issues, selection and matching issues, and issues proving the existence of solutions to integral and differential equations can all be resolved using the fundamental tools provided by fixed point theory.

For single-valued or set-valued mappings of abstract metric spaces, fixed point theorems are given. The fixed-point theorems for set-valued mappings are particularly useful in optimal control theory and have been widely applied to several issues in game theory and economics. The equation does not, necessarily have a fixed point if H is not self-mapping. In this situation, finding a rough answer x with the lowest possible error$d(x, Kx)$ is important. The best approximation theory is predicated on this assumption.

This Special Issue's goal is to report on the most recent developments in problem-solving techniques, particularly those that make use of the fixed/best-proximity point theory. Such issues are typically studied in certain function spaces to see if they can be solved. The solvability of nonlinear operator equations can be significantly influenced by the selection of the suitable fixed/best-proximity point theorem and utilisation of the unique qualities of the underlying function space. We want to give researchers a forum where they may communicate, discuss, and promote numerous fresh problems and advancements in this field.

THE BANACH CONTRACTION PRINCIPLE [R P Pant 2002]

Definition- Let X be a metric space having the distance function d. A map $h: X \to X$ is called a Lipschitz continuous if $\mu \geq 0$ exist such that

$$d\big(h(x_1), h(x_2)\big) \leq \mu\, d(x_1, x_2)\,, \qquad \forall\, x_1, x_2 \in X$$

The smallest μ that satisfies above inequality is called the Lipschitz constant of h. If $\mu \leq 1$ h is called non expensive. If $\mu < 1$ then h is called contraction.

Theorem: Assume that h be a contraction on a complete metric space h.

Then h has a unique fixed point $x' \in X$.

BROUWER'S THEORY [R P Pant 2002]

A theorem of algebraic topology in mathematics that was formulated and shown in 1912 by Dutch mathematician L.E.J. Brouwer. Brouwer looked into the behaviour of continuous functions (see continuity) by mapping the ball of unit radius in n-dimensional Euclidean space onto itself. He was inspired by earlier work by the French mathematician Henri Poincaré. A function is continuous in this context if it maps nearby points to nearby points. According to Brouwer's fixed point theorem, any such function f has at least one point x where $h(x) = x$, or where the function h maps x to itself. A fixed point of the function is what is known as such a point.

Restricted to the one-dimensional case, Brouwer's theorem can be shown to correspond to the average value theorem, which is a well-known result in calculus and states that if a continuous real-valued function h is defined on the closed interval $[-1,1]$ satisfying $h(-1) < 0$ and $h(1) > 0$, then $h(x) = 0$ for at least one number x between -1 and 1; less formally, a continuous curve passes through every value between its endpoints. The n-dimensional version of the intermediate value theorem was shown to correspond to Brouwer's fixed point theorem in 1940.

There are many other fixed-point theorems, including one for the sphere, which is the surface of a fixed sphere in three-dimensional space, for which Brouwer's theorem does not hold. The Spherical Fixed

Point Theorem states that any continuous function that maps the sphere onto itself either has a fixed point or maps some point to its antipodal point.

Fixed point theorems are examples of existence theorems because they assert the existence of objects such as solutions to functional equations, but not necessarily methods for finding such solutions. However, some of these theorems are combined with algorithms that produce solutions especially for problems in modern applied mathematics.

It is a result from topology which says that irrespective of how we stretch, twist or deform a disc if we don't tear it then in such a case there will always exist one such point which will end up being in its original position.

APPLICATION [R P Pant 2002]:

A fixed-point theorem is a result that states that under some general condition, at least one fixed point exists. According to some authors, such results are among the most widely applicable in mathematics.

Equilibria and stability are fundamental concepts in many fields that can be described using fixed points. Some examples follow.

- A fixed point of a projectivity is known as a double point in projective geometry.
- A Nash equilibrium of a game is a fixed point of the game's best response correspondence in economics. In his seminal paper that earned him the Nobel Prize in Economics, John Nash used the Kakutani fixed-point theorem.
- In physics, more specifically in phase transition theory, linearization near an unstable fixed point led to Wilson's Nobel Prize-winning work inventing the renormalization group and the mathematical explanation of the term "critical phenomenon."
- Fixed point computations are used by programming language compilers for program analysis, such as data-flow analysis, which is frequently required for code optimization. They are also the central concept of the abstract interpretation method of generic program analysis.
- The fixed-point combinator in type theory allows the definition of recursive functions in the untyped lambda calculus.
- The vector of PageRank values for all web pages represents the fixed point of a linear transformation derived from the World Wide Web's link structure. A Markov chain's stationary distribution is the fixed point of the one-step transition probability function.

Logician In his influential theory of truth, Saul Kripke employs fixed points. He demonstrates how to generate a partially defined truth predicate (one that remains undefined for problematic sentences such as "This sentence is not true") by recursively defining "truth" beginning with the segment of a language that contains no occurrences of the word and continuing until no newly well-defined sentences are produced. (This requires an infinite number of steps.) That is, for a language L, let L′ (read "L-prime") be the language generated by adding to L, for each sentence S in L, the sentence "*S is true.*" A fixed point is reached when L′ is L; at this point sentences like "*This sentence is not true*" remain undefined, so according to Kripke, the theory is appropriate for a natural language with its own truth predicate.

KANNAN TYPE FUNCTION [Ahmed Chaouki 2021]:

Definition 1. Let E be a metric space and $T : E \to E$. If there exists a number $\beta, 0 < \beta < \frac{1}{2}$, such that, for all $x, y \in E, d(T(x), T(y)) \leq \beta[d(x, T(x)) + d(y, T(y))]$ then T is called a Kannan type mapping. Note the mapping defined by Kannan may or may not continuous.

Theorem: If T1 and T2 are two mappings on metric space (E, d) into itself which complete, and if $d(T1(x), T2(y)) \leq \beta[d(x, T1(x)) + d(y, T2(y))]\ \forall x, y \in E, (1)\ where\ 0 < \beta < \frac{1}{2}$, then $T1$ and $T2$ have a unique fixed point which is common for both $T1$ and $T2$.

The next lemma plays an important role in the proof of our main theorems

Lemma: Let $Y, d)$ is a metric space and $\{x_n\}$ is a sequence in Y such that

$$d(x_n, x_{n+1}) \leq \mu_n d(x_{n-1}, x_n) \qquad (2)$$

For all $n \in N^*$ where

$$\mu_n = \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1}$$

Then $\{x_n\}$ is a Cauchy sequence.

PROBABILISTIC APPROACH [Tripathi Piyush 2016]:

Before discuss our main result, we introduce some notion of probabilistic space with distribution function.

Definition: Let L_+ be the class of distribution function $F:[0,\infty]\to[0,\infty]$ with the property,

(i) $F(0)=0,$

(ii) F is non decreasing,

and (iii) F is left continuous on $(0,\infty)$.

Suppose D_+ is subset of L_+ containing functions F such that $\lim_{x\to\infty} F(x)=1$. The specific distribution function ε_0, defined by,

$$\varepsilon_0(x) = 0 \text{ if } x = 0$$

$$= 1 \text{ if } x > 0,$$

lies in D.

Definition: A probabilistic metric space (PM space) is an ordered pair (X, F), where X is a nonempty set and $F: X \times X \to L_+$ is a mapping such that, by denoting $F(p,q)$ by $F_{p,q}$, we have,

(I) $F_{p,q}(x)=1\ \forall\ x>0$ iff $p=q$,

(II) $F_{p,q}(0)=0$,

(III) $F_{p,q}=F_{q,p}$,

and (IV) $F_{p,q}(x)=1$ and $F_{q,r}(y)=1 \Rightarrow F_{p,r}(x+y)=1$.

We note that $F_{p,q}(x)$ is value of the function $F_{p,q}=F(p,q)\in L_{+}$ at $x\in R$.

MAIN RESULT:

The contraction theory and Kannan type mapping discussed as above, prompted us to develop new mapping in probabilistic space and using it we prove fixed point result in probabilistic metric space.

Theorem: Let $(\boldsymbol{Y},\boldsymbol{F})$ be a complete probabilistic metric space. Let $k: Y \to Y$ be a mapping satisfying the condition

$$F_{kp,kq}(x) \leq \frac{F_{p,kq}(x)+F_{q,kp}(x)}{F_{p,kp}(x)+F_{q,kq}(x)+1}\, max\{F_{p,kp}(x),\ F_{q,kq}(x)\} \quad \forall\, p,q\ \in Y$$

Then k has a unique fixed point in Y.

Proof. Choose $p_0\ \in Y$ and define a sequence (p_n) such that

$$p_1 = kp_0\ , p_2 = kp_1 = k^2\, p_0\ , \ldots, p_{n+1} = kp_n =\ k^{n+1}\, p_0\ \ \forall\, n\ \in\ \mathbb{N}$$

Now

$$F_{p_n,p_{n+1}}(x) =\ F_{kp_{n-1},kp_n}(x)$$
$$\leq\ \frac{F_{p_{n-1},p_n}(x)+\ F_{p_n,p_{n+1}}(x)}{F_{p_{n-1},p_n}(x)+\ F_{p_n,p_{n+1}}(x)+1}\ max\{F_{p_{n-1},p_n}(x),\ F_{p_n,p_{n+1}}(x)\}$$
$$=\ \frac{F_{p_{n-1},p_n}(x)+\ F_{p_n,p_{n+1}}(x)}{F_{p_{n-1},p_n}(x)+\ F_{p_n,p_{n+1}}(x)+1}\ F_{p_{n-1},p_n}(x) \quad \leq\ F_{p_{n-1},p_n}(x)$$

As $n\ \to\ \infty$, it is clear that (p_n) is a Cauchy sequence, then by completeness of probabilistic space, (p_n) conveneges to some point $p\ \in Y$.

Again, $$F_{kp_n,kp}(x)\ \leq\ \frac{F_{p_n,kp}(x)+\ F_{p,p_{n+1}}(x)}{F_{p_n,p_{n+1}}(x)+\ F_{p,kp}(x)+1}\ max\{F_{p_n,p_{n+1}}(x),\ F_{p,kp}(x)\}$$

Taking $n\ \to\ \infty$, we get

$$F_{p,kp}(x)\ \leq\ \frac{F_{p,kp}(x)}{F_{p,kp}(x)+1}\ \leq\ F_{p,kp}(x)$$

Therefore $F_{p,kp}(x) = 0\ \Longrightarrow kp = p.$

Therefore p is a fixed point of k.

For uniqueness, suppose q is another fixed point of k in Y.

Then $F_{p,q}(x)\ \leq\ F_{kp,kq}(x)\ \leq 0$ hence $p = q$.

References.

Ahmed Chaouki Aouiney and Abdelkrim Aliouchez: "Fixed Point Theorems Of Kannan Type With An Application To Control Theory", *Applied Mathematics E-Notes*, 21, (2021), pp. 238-249.

Pant R.P., Lohani A. B. and Jha K.: "A HISTORY OF FIXED POINT THEOREMS", *GANITA BHARATI*, 24 (14),(2002), pp. 147 -l 59.

Tripathi Piyush, Mishra S N and Gupta Manisha: "Application of Contraction Mapping In Menger Spaces", Global Journal of Pure and Applied Mathematics. Volume 12, Number 2 (2016), pp. 1629-1634