STRONG PERFECT NONBONDAGE NUMBER OF SOME GRAPHS

Govindalakshmi T.S[1*], Meena N[2]

Abstract
Let G be a simple graph. A subset S ⊆ V(G) is called a strong (weak) perfect dominating set of G if $|N_s(u) \cap S| = 1$ ($|N_w(u) \cap S| = 1$) for every u ∈V(G) - S where $N_s(u) = \{v \in V(G)/uv \in E(G), \deg v \geq \deg u\}$ ($N_w(u) = \{v \in V(G)/uv \in E(G), \deg v \leq \deg u\}$. The minimum cardinality of a strong (weak) perfect dominating set of G is called the strong (weak) perfect domination number of G and is denoted by $\gamma_{sp}(G)(\gamma_{wp}(G))$. The strong perfect non bondage number $b_{spn}(G)$ of a nonempty graph G is defined as the maximum cardinality among all sets of edges X ⊆ E(G) for which $\gamma_{sp}(G - X) = \gamma_{sp}(G)$. If $b_{spn}(G)$ does not exist, then $b_{spn}(G)$ is defined as zero. In this paper strong perfect nonbondage number of some standard graphs are determined.

Keywords: Strong perfect dominating set, strong perfect domination number and strong perfect nonbondage number.

[1*]Research Scholar, Reg. No. 12566, P.G. & Research Department of Mathematics, The M.D.T. Hindu College, Tirunelveli –10 & Assistant professor of Mathematics, Manonmaniam Sundaranar University College, Puliangudi-627855. Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli -627 012, TamilNadu, India. Email: laxmigopal2005@gmail.com
[2]Assistant professor of Mathematics, P.G. & Research Department of Mathematics, The M.D.T. Hindu College, Tirunelveli-10, Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627 012, Tamil Nadu, India. Email: meena@mdthinducollege.org

*C**orresponding Author:** Govindalakshmi T.S
*Research Scholar, Reg. No. 12566, P.G. & Research Department of Mathematics, The M.D.T. Hindu College, Tirunelveli –10 & Assistant professor of Mathematics, Manonmaniam Sundaranar University College, Puliangudi-627855. Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli -627 012, TamilNadu, India. Email: laxmigopal2005@gmail.com

AMS subject classification: 05C69

DOI: 10.48047/ecb/2023.12.si10.00310

1.INTRODUTION

By a graph, it is meant that a finite, undirected graph without loops and multiple edges. Let G be a graph with vertex set V(G) and edge set E(G). The neighbourhood of v, written as N(v) is defined by N(v) = {uϵV(G)/ u is adjacent to v}. The degree of any vertex u in G is the number of edges incident with u and is denoted by deg u. The minimum and maximum degrees of vertices in G are denoted by $\delta(G)$ and $\Delta(G)$ respectively. A dominating set D of G is a subset of V(G) such that every vertex in V – D is adjacent to at least one vertex in D. A dominating set of G [9,10] of minimum cardinality is a minimum dominating set of G and cardinality is the domination number of G. It is denoted by $\gamma(G)$. A set $D \subseteq V(G)$ is a strong dominating set of G [8] if every vertex in V – D is strongly dominated by at least one vertex in D. Similarly, D is a weak dominating set if every vertex in V – D is weakly dominated by at least one vertex in D. The strong (weak) domination number $\gamma_s(G)(\gamma_w(G))$ is the minimum cardinality of a strong (weak) dominating set of G. A dominating set S is a perfect dominating set of G [1,2] if $|N(v) \cap S| = 1$ for each $v \in V - S$. Minimum cardinality of the perfect dominating set of G is the perfect domination number of G [7] and it is denoted by $\gamma_p(G)$. Motivated by these definitions, the strong perfect domination in graph was introduced by T.S. Govindalakshmi and N. Meena [4]. In [6], Kulli and Janakiram introduced the concept of nonbondage number as follows. The nonbondage number $b_n(G)$ of a nonempty graph G is the maximum cardinality among all sets of edges $X \subseteq E(G)$ for which $\gamma_{sp}(G - X) = \gamma_{sp}(G)$ for an edge set X. X is called the nonbondage set and the maximum one is the maximum nonbondage set. In this paper strong perfect nonbondage number of a graph is defined and strong perfect nonbondage number of standard graphs are determined. For all graph theoretic terminologies and notations Harary [5] is followed.

Definition 2.1[4] Let G be a simple graph. A subset $S \subseteq V(G)$ is called a strong (weak) perfect dominating set of G if $|N_s(u) \cap S| = 1$ ($|N_w(u) \cap S| = 1$) for every $u \in V(G) - S$ where $N_s(u) = \{v \in V(G)/uv \in E(G), \deg v \geq \deg u\}$ ($N_w(u) = \{v \in V(G)/ uv \in E(G), \deg v \leq \deg u\}$.

Remark 2.2.[4] The minimum cardinality of a strong (weak) perfect dominating set of G is called the strong (weak) perfect domination number of G and is denoted by $\gamma_{sp}(G)(\gamma_{wp}(G))$.

Definition 2.3. Bi star is the graph obtained by joining the apex vertices of two copies of star $K_{1,n}$.

Definition 2.4. The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 (where G_i has p_i points and q_i lines) is defined as the graph G obtained by taking one copy of G_1 and p_1 copies of G_2, and then joining by a line the i^{th} point of G_1 to every point in the i^{th} copy of G_2.

Definition 2.5. The wheel W_n is defined to be the graph $C_{n-1} + K_1$, $n \geq 4$.

Definition 2.6. The helm H_n is the graph obtained from the wheel W_n with n spokes by adding n pendant edges at each vertex on the wheel's rim.

Theorem 2.7. [4] For any path P_m**,**

Then $\gamma_{sp}(P_m) = \begin{cases} n \ if \ m = 3n, n \epsilon N \\ n+1 \ if \ m = 3n+1, n \epsilon N \\ n+2 \ if \ m = 3n+2, n \ \epsilon N \end{cases}$

Theorem 2.8. [4] For any cycle C_m**,**

Then $\gamma_{sp}(C_m) = \begin{cases} n \ if \ m = 3n, n \epsilon N \\ n+1 \ if \ m = 3n+1, n \epsilon N \\ n+2 \ if \ m = 3n+2, n \ \epsilon N \end{cases}$

Theorem 2.9. [4] Let G be a connected graph with $|V(G)| = n$. Then $\gamma_{sp}(G \odot K_1) = n$.

Remark 2.10. [4]

(i) $\gamma_{sp}(D_{r,s}) = 2$. r, s ϵ N

(ii) $\gamma_{sp}(K_n) = 1$.

(iii) $\gamma_{sp}(K_{1,n}) = 1$.

(iv) $\gamma_{sp}(W_n) = 1$.

(v) $\gamma_{sp}(H_n) = n$. $n \geq 5$ *and* $\gamma_{sp}(H_4) = 3$.

3. MAIN RESULTS

Definition3.1: The strong perfect nonbondage number of G denoted $b_{spn}(G)$, is defined as the maximum cardinality among all sets of edges $X \subseteq E(G)$ forwhich $\gamma_{sp}(G - X) = \gamma_{sp}(G)$. If $b_{spn}(G)$ does not exist, then $b_{spn}(G)$ is defined as zero.

Example 3.2: Consider the graph $G = C_6 \odot K_1$ in figure 1, $\gamma_{sp}(G) = 6$.

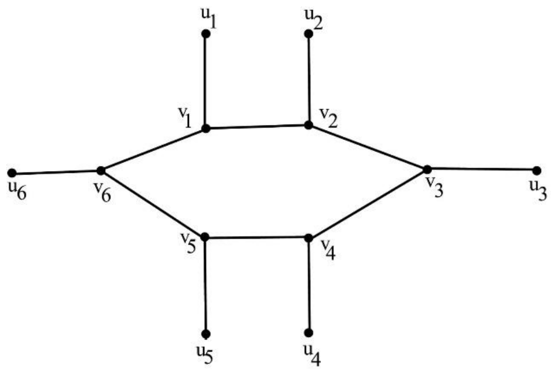

Figure 1, Graph $C_6 \odot K_1$

If anyone edge of the cycle is removed from G then the new graph G' is $P_6 \odot K_1$. Therefore $\gamma_{sp}(G') = 6 = \gamma_{sp}(G)$. If anyone edge v_iu_i, $1 \leq i \leq 6$ is removed then $\gamma_{sp}(G') = 6 = \gamma_{sp}(G)$. If any two edges of the cycle are removed from G then G' is $P_2 \cup (P_5 \odot K_1)$ or $2(P_3 \odot K_1)$ or $(P_2 \odot K_1) \cup (P_4 \odot K_1)$. Therefore $\gamma_{sp}(G') = 6 = \gamma_{sp}(G)$. If any two edges v_iu_i, $1 \leq i \leq 6$, are removed from G then also $\gamma_{sp}(G') = 6 = \gamma_{sp}(G)$. If one edge v_iu_i, $1 \leq i \leq 6$ and one edge from the cycle are removed from G then $\gamma_{sp}(G') = 6 = \gamma_{sp}(G)$. Let $X = \{v_1v_2, v_4v_5, v_3u_3\}$. $G - X = (P_3 \odot K_1) \cup P_5 \cup P_1$. Therefore $\gamma_{sp}(G - X) = 7 > 6 = \gamma_{sp}(G)$. Hence $b_{spn}(G) = 2$.

Observation 3.3: Let G be a graph with unique full degree vertex v. $\gamma_{sp}(G) = 1$. If any edge incident with v is removed then strong perfect domination number of the resulting graph G' is greater than or equal to 2. Hence $b_{spn}(G) = 0$.

Observation 3.4: $b_{spn}(K_{1,n}) = 0$, $n \geq 1$.

Theorem 3.5: For any path P_m on m vertices, $b_{spn}(P_m) = 0$, $m \geq 2$ and $m \neq 4$.

Proof: Let $G = P_m$, $m \geq 2$. Let $V(G) = \{v_i / 1 \leq i \leq m\}$

Case (1): Let m = 3n, $n \geq 1$. Let $X = \{v_1v_2\}$ or $\{v_{n-1}v_n\}$. $G - X = P_1 \cup P_{3n-1} = P_1 \cup P_{3(n-1)+2}$. Therefore $\gamma_{sp}(G - X) = n+2 > \gamma_{sp}(G)$. Hence $b_{spn}(G) = 0$.

Case (2): Let m = 3n+1, $n \geq 2$. Let $X = \{v_2v_3\}$. $G - X = P_2 \cup P_{3n-1} = P_2 \cup P_{3(n-1)+2}$. Therefore $\gamma_{sp}(G - X) = n+2 > \gamma_{sp}(G)$. Hence $b_{spn}(G) = 0$.

Case (3): Let m = 3n+2, $n \geq 1$. Let $X = \{v_2v_3\}$. $G - X = P_2 \cup P_{3n}$. Therefore $\gamma_{sp}(G - X) = n+1 < \gamma_{sp}(G)$. Hence $b_{spn}(G) = 0$.

Remark 3.6: Let m = 4. $X = \{v_1v_2\}$ or $\{v_2v_3\}$ or $\{v_3v_4\}$. $G - X = P_1 \cup P_3$ or $2P_2$. Therefore $\gamma_{sp}(G - X) = 2 = \gamma_{sp}(G)$. Remove any two edges. Then the resulting graph G′ is $2P_1 \cup P_2$. Therefore $\gamma_{sp}(G') = 3 > \gamma_{sp}(G)$. Hence $b_{spn}(P_4) = 1$.

Theorem3.7: For any cycle C_m on m vertices, $b_{spn}(C_m) = 1$, $m \geq 3$ and $m \neq 4$.

Proof: Let $G = C_m$, $m \geq 3$. Let $V(G) = \{u_i / 1 \leq i \leq m\}$.

Case (1): Let m = 3n, $n \geq 1$. If anyone edge is removed from the cycle G then the resulting graph G' is P_{3n}. Therefore $\gamma_{sp}(G') = n = \gamma_{sp}(G)$. Remove two edges from G such that the resulting graph G' is $P_2 \cup P_{3n-2}$. Therefore $\gamma_{sp}(G') = \gamma_{sp}(P_2) + \gamma_{sp}(P_{3(n-1)+1}) = n + 1 > \gamma_{sp}(G)$. Hence $b_{spn}(G) = 1$.

Case (2): Let m = 3n+1, $n \geq 2$. If anyone edge is removed from the cycle G then the resulting graph G' is P_{3n+1}. Therefore $\gamma_{sp}(G') = n+1 = \gamma_{sp}(G)$. Remove two edges from G such that the resulting graph G' is $P_2 \cup P_{3n-1}$. Therefore $\gamma_{sp}(G') = \gamma_{sp}(P_2) + \gamma_{sp}(P_{3(n-1)+2}) = n + 2 > \gamma_{sp}(G)$. Hence $b_{spn}(G) = 1$.

Case (3): Let m = 3n+2, $n \geq 1$. If anyone edge is removed from the cycle G then the resulting graph G' is P_{3n+2}. Therefore $\gamma_{sp}(G') = n+2 = \gamma_{sp}(G)$. Remove two edges from G such that the resulting graph G' is $P_2 \cup P_{3n}$. Therefore $\gamma_{sp}(G') = \gamma_{sp}(P_2) + \gamma_{sp}(P_{3n}) = n + 1 < \gamma_{sp}(G)$. Hence $b_{spn}(G) = 1$.

Remark 3.8: Let m = 4. If anyone edge is removed from the cycle C_4 then the resulting graph G' is P_4. Therefore $\gamma_{sp}(G') = 2 = \gamma_{sp}(G)$. If any two edges are removed from the cycle C_4 then the resulting graph G' is $P_1 \cup P_3$ or $2P_2$.Therefore $\gamma_{sp}(G') = 2 = \gamma_{sp}(G)$. If any three edges are removed from the cycle C_4 then the resulting graph G' is $2P_1 \cup P_2$. Therefore $\gamma_{sp}(G') = 3 > \gamma_{sp}(G)$. Hence $b_{spn}(G) = 2$.

Theorem 3.9: For any complete graph K_n on n vertices,

$$b_{spn}(K_n) = \begin{cases} \frac{n}{2} - 1 & if\ n\ is\ even\ and\ n \geq 2 \\ \frac{n-1}{2} & if\ n\ is\ odd\ and\ n \geq 3 \end{cases}$$

Proof: Let $G = K_n$, $n \geq 3$ and $n \neq 4$. $V(G) = \{v_i / 1 \leq i \leq n\}$. $\gamma_{sp}(K_n) = 1$. All the vertices of K_n are full degree vertices. To increase the strong perfect domination number of G, degree of each vertex must be reduced by at least one.

Case (1): Suppose n is even and n≥ 2. Let X = $\{v_iv_{i+1}/1 \le i \le n-1$ and i is odd$\}$. $|X| = \frac{n}{2}$. deg v_i = $n-2$, $1 \le i \le n$. $\{v_i /1 \le i \le n\}$ is the unique strong perfect dominating set of G – X. Hence $\gamma_{sp}(G - X) = n > \gamma_{sp}(G)$. Remove $\frac{n}{2} - 1$ edges from G such that atleast one full degree vertex exist in G – X. Also, removal of no set of less than $\frac{n}{2} - 1$ edges increase the strong perfect domination number of the resulting graph. Hence $b_{spn}(G) = \frac{n}{2} - 1$.

Case (2): Suppose n is odd and n≥ 3. As in case (1), if the degree of each vertex is reduced by at least one then strong perfect domination number of the resulting graph increases. Remove at least $\frac{n+1}{2}$ edges so that the resulting graph G' has no full degree vertex. Therefore $\gamma_{sp}(G') > \gamma_{sp}(G)$. Hence remove $\frac{n+1}{2} - 1 = \frac{n-1}{2}$ edges from G such that atleast one full degree vertex exist in G – X. Also, removal of no set of less than $\frac{n-1}{2}$ edges increase the strong perfect domination number of the resulting graph. Hence $b_{spn}(G) = \frac{n-1}{2}$.

Theorem 3.10: For any complete bipartite graph $K_{m,n}$ on m + n vertices,

$$b_{spn}(K_{m,n}) = \begin{cases} n-1 & if\ m = n \\ mn - n - m^2 + m - 1 & if\ n > m \end{cases}$$

Proof: Let G = $K_{m,n}$, m, n ≥ 1. V(G) = $\{v_i, u_j /1 \le i \le m, 1 \le j \le n\}$ and E(G) = $\{v_iu_j/1 \le i \le m, 1 \le j \le n\}$. $\gamma_{sp}(K_{m,n}) = 2$ if m = n and m + n if m ≠ n.

Case (1): Let m = n. deg v_i = deg u_i = n, $1 \le i \le n$. Removal of no set of less than n edges increase the strong perfect domination number of the resulting graph. Let X = $\{v_iu_j/ 1 \le j \le n\}$ for some i = 1 to n. G – X = $K_{n-1,n} \cup K_1$. $\gamma_{sp}(G - X) = 2n > 2 = \gamma_{sp}(G)$. Hence $b_{spn}(G) = n - 1$.

Case (2): Let n > m. Remove (m – 1) (n – m) edges from G such that deg v_i = n, for any i = 1 to n, deg u_j = m, deg v_t = m, $1 \le t \le n$, t ≠i and deg u_k < n, $1 \le k \le n$, $k \ne j$. $\{v_i, u_j\}$ is the unique strong perfect dominating set of the resulting graph G'. $\gamma_{sp}(G') = 2 < \gamma_{sp}(G)$. Hence $b_{spn}(G) \le mn - n - m^2 + m - 1$. Removal of no set of greater than (n – m) (m – 1) edges decreases the strong perfect domination number of the resulting graph. Hence $b_{spn}(G) = mn - n - m^2 + m - 1$.

Theorem 3.11: For any bistar $D_{r,s}$ on r+s+2 vertices, $b_{spn}(D_{r,s}) = 0$, r ≥ s, r, s≥1.

Proof: Let G = $D_{r,s}$, r, s ≥ 1. Let V(G) = $\{u, v, u_i, v_j/ 1 \le i \le r, 1 \le j \le s\}$. $\gamma_{sp}(G) = 2$. Let r ≥ s, r, s≥1. Let e = uv. G – e = $K_{1,r} \cup K_{1,s}$. Therefore $\gamma_{sp}(G - e) = 2 = \gamma_{sp}(G)$. Let e = uu_i, $1 \le i \le r$ or vv_j, $1 \le j \le s$. G – e = $K_1 \cup D_{r-1,s}$ or $K_1 \cup D_{r,s-1}$ Therefore $\gamma_{sp}(G - e) = 3 > \gamma_{sp}(G)$. Hence $b_{spn}(G) = 0$.

Theorem 3.12: For any helm H_n, n ≥4,

$$b_{spn}(H_n) = \begin{cases} 2 & if\ n = 4 \\ 0 & if\ n = 5 \\ 1 & if\ n \ge 6 \end{cases}$$

Proof: Let G = H_n, n≥4. V(G)= $\{v, v_i, u_i /1 \le i \le n-1\}$ and E(G) = $\{vv_i, v_iu_i/1 \le i \le n-1\} \cup \{v_iv_{i+1}/ 1 \le i \le n-2\} \cup \{v_{n-1}v_1\}$. $\gamma_{sp}(G) = n$, n≥5. deg v_i = 4, deg u_i = 1, $1 \le i \le n-1$, deg v = n-1. $\gamma_{sp}(G) = 3$ when n = 4.

Case (1): Let n = 4. If any edge is removed from G, obviously $\gamma_{sp}(G) = 3$. Hence $b_{spn}(G) \ge 1$. Removal of any two edges from G does not affect the strong perfect domination number of the resulting graph. Let X = $\{v_1v_2, v_2v_3, v_3v_1\}$. $\{v, u_1, u_2, u_3\}$ is the unique strong perfect dominating set of G – X. Therefore $\gamma_{sp}(G - X) = 4 > \gamma_{sp}(G)$. Hence $b_{spn}(G) = 2$.

Case (2): Let n = 5. $\gamma_{sp}(H_5) = 5$. Let e = vv_i, $1 \le i \le n-1$. Let X = {e}. Let v_k be the vertex not adjacent with v_i. S = $\{v_k, u_j, v_i/1 \le j \le n-1, j \ne i\}$ is the unique strong perfect dominating set of G – X. |S| = 4. Therefore $\gamma_{sp}(G - X) = 4 < \gamma_{sp}(G)$. Hence $b_{spn}(G) = 0$.

Case (3): Let n ≥ 6. Suppose any edge vv_i, $1 \le i \le n-1$ is removed from G. $\{v, v_i, u_j/1 \le j \le n-2, j \ne i\}$ is the unique strong perfect dominating set of the resulting graph G′. Therefore $\gamma_{sp}(G') = n = \gamma_{sp}(G)$. Suppose any edge v_iv_{i+1}, $1 \le i \le n-2$, or $v_{n-1}v_1$ or v_iu_i, $1 \le i \le n-1$ is removed from G. $\{v, u_i/1 \le i \le n-1\}$ is the unique strong perfect dominating set of the resulting graph G'. Therefore $\gamma_{sp}(G') = n = \gamma_{sp}(G)$. Hence $b_{spn}(G) \ge 1$. Let X = $\{vv_i, v_iu_i\}$, $1 \le i \le n-1$. $\{v, v_i, u_j, u_i/1 \le j \le n-2, i \ne j\}$ is the unique strong perfect dominating set of G – X. Therefore $\gamma_{sp}(G - X) = n + 1 > \gamma_{sp}(G)$. Hence $b_{spn}(G) \le 1$. Hence $b_{spn}(G) = 1$.

Observation 3.13: Since $W_4 = K_4$, $b_{spn}(W_4) = 1$. W_n, n≥5 has unique full degree vertex, $b_{spn}(W_n) = 0$.

Reference

1. Chaluvaraju. B, 2010, Perfect k-domination in graphs, Australasian Journal of Combinatorics, Vol. 48, pp 175 – 184.
2. Changela J.V & Vala G.J, 2014, Perfect domination and Packing of a graph, International Journal of Engineering and Innovative Technology (IJEIT) Volume 3, Issue 12, p61- 64.
3. Ebadi. K and PushpaLathaL, 2010, The Strong Nonbondage Number of a Graph, International Mathematical Forum, 5, No.34, 1691-1696.
4. Govindalakshmi T.S & Meena N, 2018, Strong Perfect Domination in Graphs, International Journal of Mathematics Trends and Technology, Volume 58, Issue 3, p29-33.
5. Harary.F, 1969, Graph Theory, Addison-Wesley.
6. Kulli V. R & Janakiram B, 1996, The Nonbondage number of a graph, Graph Theory Notes of Newyork, New York Academy of Science, XXX 14-16.
7. Livingston .M & Stout. Q.F, 1990, Perfect dominating sets. Congr.Numer.,79: 187-203.
8. Sampathkumar E & PushpaLatha L,1996, Strong weak domination and domination balance in a graph, Discrete Math., 161:235-242.
9. Teresa W. Haynes, Stephen T. Hedetniemi, Peter J. Slater (Eds), 1998, Domination in graphs: Advanced Topics, Marcel Decker, Inc., New York.
10. Teresa W. Haynes, Stephen T. Hedetniemi, Peter J. Slater, 1998, Fundamentals of domination in graphs, Marcel Decker, Inc., New York.