Stalk spaces, Stalk maps and Fixed points

Ch.Srinivasa Rao[1], Kurada Chinni Naga Tarani[2*], Nistala. V. E. S. Murthy[3]

Abstract: A new concept of stalk spaces, stalk maps and establish fixed point theorems in stalk spaces for stalk maps is introduced. The results established are illustrated through several examples to give a good insight. We also introduce the notion of continuity of stalk maps. We introduce a new contractive condition namely ST-condition for stalk maps.

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[1,2,3]Departmentof Mathematics, Andhra University, Visakhapatnam-530003, India.
[1]Email: drcsr41@gmail.com
[2*]Corresponding Author Email: kuradatarani627@gmail.com
3Email: drnvesmurthy@rediffmail.com

I. INTRODUCTION

Metric fixed point theory is widely studied by many authors such as [**1**], [**2**], [**3**], [**4**], [**5**] and [**6**]. Finding sufficient conditions for the existence and uniqueness of fixed points for self maps on various spaces such as metric spaces, b-metric spaces [2], 2-metric spaces [**4**], fuzzy metric spaces [**5**], [**6**] and bipolar metric spaces [**8**], [**7**] is extensively studied.

Various contraction conditions are imposed on self maps to arrive at a fixed point constructively.

Now we propose a new concept of stalk spaces and stalk maps on such spaces and obtain conditions for stalk maps to have only one mother (fixed point).

II. Stalk spaces, Stalk maps and Examples

Definition 2.1. Let Θ, Ω and Ψ be three non-empty sets such that $\Theta \cap \Omega = \Omega \cap \Psi = \Psi \cap \Theta = \Theta \cap \Omega \cap \Psi$. Suppose $\varrho: \Theta \times \Omega \times \Psi \to [0, \infty)$satisfies the following conditions:

(2.1.1). If $\theta, \omega, \psi \in \Theta \cap \Omega \cap \Psi$ then $\varrho(\theta, \omega, \psi) = 0iff\theta = \omega = \psi$.

(2.1.2). $\varrho(\theta, \omega, \psi) = \varrho(u, v, w)$ for any permutation $\{u, v, w\}$ of $\{\theta, \omega, \psi\}$ whenever $\theta, \omega, \psi \in \Theta \cap \Omega \cap \Psi$.

(2.1.3).$|\varrho(\theta_1, \omega_1, \psi_1) - \varrho(\theta_2, \omega_2, \psi_2)| \leqslant$

$$\min\left\{\varrho(\theta_1, \omega_1, \psi_2)\varrho(\theta_2, \omega_2, \psi_1), \varrho(\theta_2, \omega_1, \psi_2), \varrho(\theta_2, \omega_1, \psi_1) + \varrho(\theta_1, \omega_2, \psi_2)\right\}$$

for all $(\theta_1, \omega_1, \psi_1), (\theta_2, \omega_2, \psi_2) \in \Theta \times \Omega \times \Psi$.

Then $(\boldsymbol{\Theta}, \boldsymbol{\Omega}, \boldsymbol{\Psi}, \boldsymbol{\varrho})$ is called a stalk space and $\boldsymbol{\varrho}$ is called stalk metric.

In this context Θ, Ω, Ψ are called stalks of $\Theta \cup \Omega \cup \Psi$ and $\Theta \cap \Omega \cap \Psi$ is called core. Further $\Theta \cup \Omega \cup \Psi$ is called bundle of three stalks, namely, Θ, Ω and Ψ with core $\Theta \cap \Omega \cap \Psi$. $\varrho$is called the binder of the bundle. Members of the core are called seeds. i.e., $\ddot{a}$ is a seed if $\ddot{a} \in \Theta \cap \Omega \cap \Psi$.

Definition 2.2. A sequence $\{(\theta_n, \omega_n, \psi_n)\}$ in $\Theta \times \Omega \times \Psi$ is said to converge to $(\ddot{a}, \ddot{a}, \ddot{a}) \in$ core if $\varrho(\theta_n, \ddot{a}, \ddot{a}) \to 0, \varrho(\ddot{a}, \omega_n, \ddot{a}) \to 0$, $\varrho(\ddot{a}, \ddot{a}, \psi_n) \to 0, \varrho(\theta_n, \omega_n, \ddot{a}) \to 0$, $\varrho(\theta_n, \ddot{a}, \psi_n) \to 0$ and $\varrho(\ddot{a}, \omega_n, \psi_n) \to 0$. In this case, we write $(\theta_n, \omega_n, \psi_n) \to (\ddot{a}, \ddot{a}, \ddot{a})$.

Definition 2.3.Complete stalk space: Whenever $\varrho(\theta_n, \omega_n, \psi_n) \to 0$ there exists $\ddot{a} \in$ core $\Theta \cap \Omega \cap \Psi$such that $(\theta_n, \omega_n, \psi_n) \to (\ddot{a}, \ddot{a}, \ddot{a})$ i.e.,$\varrho(\theta_n, \ddot{a}, \ddot{a}) \to 0$, $\varrho(\ddot{a}, \omega_n, \ddot{a}) \to 0, \varrho(\ddot{a}, \ddot{a}, \psi_n) \to 0$, $\varrho(\theta_n, \omega_n, \ddot{a}) \to 0$, $\varrho(\theta_n, \ddot{a}, \psi_n) \to 0$and $\varrho(\ddot{a}, \omega_n, \psi_n) \to 0$. Then we say that the stalk space is complete.

Lemma 2.4. Let $(\Theta, \Omega, \Psi, \varrho)$ be a stalk space. If $\{(\theta_n, \omega_n, \psi_n)\} \to (\ddot{a}, \ddot{a}, \ddot{a})$ and $\{(\theta_n, \omega_n, \psi_n)\} \to (\ddot{b}, \ddot{b}, \ddot{b})$then$\ddot{a} = \ddot{b}$

Proof. We have by Definition-2.3,

$$\varrho(\ddot{b}, \ddot{b}, \ddot{a}) \leqslant \varrho(\ddot{b}, \ddot{b}, \psi_n) + \varrho(\theta_n, \omega_n, \psi_n) + \varrho(\theta_n, \omega_n, \ddot{a}) \to 0 \text{as } n \to \infty.$$

(since by Definition-2.2), $\varrho(\theta_n, \omega_n, \psi_n) \to 0$)

$\therefore \quad (\ddot{b}, \ddot{b}, \ddot{a}) = 0$and hence$\ddot{b} = \ddot{a}$.

Notation: 1. Suppose $\Theta = (\mathbb{R}, 0,0)$, $\Omega = (0, \mathbb{R}, 0)$and $\Psi = (0,0, \mathbb{R})$.

If $\ddot{a} \in \Theta \cup \Omega \cup \Psi$ then

$$\hat{\ddot{a}} = \begin{cases} 0 \text{ if } \ddot{a} = (0,0,0) \\ \theta \text{ if } \ddot{a} = (\theta, 0,0) \\ \omega \text{ if } \ddot{a} = (0, \omega, 0) \\ \psi \text{ if } \ddot{a} = (0,0, \psi) \end{cases}$$

$\hat{\ddot{a}} = 0$iff $\ddot{a} = (0,0,0)$.

2. For $\ddot{a}, \ddot{b} \in \Theta \cup \Omega \cup \Psi$

$$\| \ddot{a} - \ddot{b} \|_1 = \begin{cases} |\hat{\ddot{a}}| + |\hat{\ddot{b}}| & \text{if } \ddot{a}, \ddot{b} \text{ are in different stalks} \\ |\hat{\ddot{a}} - \hat{\ddot{b}}| & \text{if } \ddot{a}, \ddot{b} \text{ belong to the same stalk.} \end{cases}$$

$$\| \ddot{a} - \ddot{b} \|_2 = \text{The Euclidean distance between } \ddot{a} \text{and } \ddot{b}$$

$$= \begin{cases} \sqrt{\hat{\ddot{a}}^2 + \hat{\ddot{b}}^2} & \text{if } \ddot{a}, \ddot{b} \text{ are in different stalks} \\ |\hat{\ddot{a}} - \hat{\ddot{b}}| & \text{if } \ddot{a}, \ddot{b} \text{ belong to the same stalk.} \end{cases}$$

$$\varrho_0(\theta, \omega, \psi) = \| \theta - \psi \|_1 + \| \omega - \psi \|_1 + \| \theta + \omega - 2\psi \|_1$$

3. $\varrho_1(\theta, \omega, \psi) = |\hat{\theta}| + |\hat{\omega}| + |\hat{\psi}|$

4. $\varrho_2(\theta, \omega, \psi) = \sqrt{\hat{\theta}^2 + \hat{\omega}^2 + \hat{\psi}^2}$.

5. $\varrho_\infty(\theta, \omega, \psi) = \max\{|\hat{\theta}|, |\hat{\omega}|, |\hat{\psi}|\}$if $(\theta, \omega, \psi) \in \Theta \times \Omega \times \Psi$.

Example 2.5. Let $\Theta = (\mathbb{R}^+, 0,0)$, $\Omega = (0, \mathbb{R}^+, 0)$and $\Psi = (0,0, \mathbb{R}^+)$. Then $\Theta \cap \Omega \cap \Psi = \{(0,0,0)\}$.

Let $\varrho_0: \Theta \times \Omega \times \Psi \to [0, +\infty)$ be defined by

$$\varrho_0(\theta, \omega, \psi) = \| \theta - \psi \|_1 + \| \omega - \psi \|_1 + \| \theta + \omega - 2\psi \|_1.$$

Then $(\Theta, \Omega, \Psi, \varrho_0)$ is a complete stalk space.

(i) $\theta, \omega, \psi \in$core then clearly $\theta = \omega = \psi = (0,0,0)$.

Hence (2.1.1) is satisfied.

(ii) Clearly (2.1.2) is satisfied.

(iii) Suppose $(\theta_1, \omega_1, \psi_1), (\theta_2, \omega_2, \psi_2) \in \Theta \times \Omega \times \Psi$, we show that

$$\varrho_0(\theta_1, \omega_1, \psi_1) \leqslant \varrho_0(\theta_1, \omega_1, \psi_2) + \varrho_0(\theta_2, \omega_2, \psi_2) + \varrho_0(\theta_2, \omega_2, \psi_1)$$

We observe that

$$\varrho_0(\theta, \omega, \psi) = \| \theta - \psi \|_1 + \| \omega - \psi \|_1 + \| \theta + \omega - 2\psi \|_1 = 2|\hat{\theta}| + 2|\hat{\omega}| + 4|\hat{\psi}|$$

From this we have

$$\varrho_0(\theta_1, \omega_1, \psi_1) = 2|\hat{\theta}_1| + 2|\hat{\omega}_1| + 4|\hat{\psi}_1|,$$

$$\varrho_0(\theta_1, \omega_1, \psi_2) = 2|\hat{\theta}_1| + 2|\hat{\omega}_1| + 4|\hat{\psi}_2|$$

$$\varrho_0(\theta_2, \omega_2, \psi_2) = 2|\hat{\theta}_2| + 2|\hat{\omega}_2| + 4|\hat{\psi}_2|$$

and$\varrho_0(\theta_2, \omega_2, \psi_1) = 2|\hat{\theta}_2| + 2|\hat{\omega}_2| + 4|\hat{\psi}_1|$

$$\begin{aligned} \therefore \varrho_0(\theta_1, \omega_1, \psi_2) + \varrho_0(\theta_2, \omega_2, \psi_2) + \varrho_0(\theta_2, \omega_2, \psi_1) &= 2|\hat{\theta}_1| + 2|\hat{\omega}_1| + 4|\hat{\psi}_2| \\ &\quad + 2|\hat{\theta}_2| + 2|\hat{\omega}_2| + 4|\hat{\psi}_2| \\ &\quad + 2|\hat{\theta}_2| + 2|\hat{\omega}_2| + 4|\hat{\psi}_1| \\ &= \varrho_0(\theta_1, \omega_1, \psi_1) + 2\varrho_0(\theta_2, \omega_2, \psi_2) \end{aligned}$$

$$\therefore \varrho_0(\theta_1, \omega_1, \psi_1) \leqslant \varrho_0(\theta_1, \omega_1, \psi_2) + \varrho_0(\theta_2, \omega_2, \psi_2) + \varrho_0(\theta_2, \omega_2, \psi_1).$$

The following can be prove similarly,

$$\varrho_0(\theta_1 \omega_1, \psi_1) \leqslant \varrho_0(\theta_1, \omega_2, \psi_1) + \varrho_0(\theta_2, \omega_1, \psi_2) + \varrho_0(\theta_2, \omega_2, \psi_2)$$

$$\varrho_0(\theta_1 \omega_1, \psi_1) \leqslant \varrho_0(\theta_2, \omega_1, \psi_1) + \varrho_0(\theta_1, \omega_2, \psi_2) + \varrho_0(\theta_2, \omega_2, \psi_1)$$

$$\varrho_0(\theta_2, \omega_2, \psi_2) \leqslant \varrho_0(\theta_1, \omega_1, \psi_2) + \varrho_0(\theta_1, \omega_1, \psi_1) + \varrho_0(\theta_2, \omega_2, \psi_1)$$

$$\varrho(\theta_2, \omega_2, \psi_2) \leqslant \varrho(\theta_1, \omega_2, \psi_1) + \varrho(\theta_1, \omega_1, \psi_1) + \varrho(\theta_2, \omega_1, \psi_2)$$

and$\varrho_0(\theta_2, \omega_2, \psi_2) \leqslant \varrho_0(\theta_2, \omega_1, \psi_1) + \varrho_0(\theta_1, \omega_1, \psi_1) + \varrho_0(\theta_1, \omega_1, \psi_2)$

From this we obtain (2.1.3)

$\therefore$ $(\Theta, \Omega, \Psi, \varrho_0)$is a stalk space.

Suppose $(\theta_n, \omega_n, \psi_n) \in \Theta \times \Omega \times \Psi$ and $\varrho_0(\theta_n, \omega_n, \psi_n) \to 0$
Since $\{(0,0,0)\}$ is the only seed in core, by the Definition-2.3,
$(\theta_n, \omega_n, \psi_n) \to (0,0,0)$as $n \to \infty$.
$\therefore$ $(\Theta, \Omega, \Psi, \varrho_0)$is a complete stalk space.

Example 2.6. Write $\Theta = (\mathbb{R}^+, 0,0), \Omega = (0, \mathbb{R}^+, 0)$ and $\Psi = (0,0, \mathbb{R}^+)$in this case we say that$\Theta, \Omega$ and Ψ are independent copies of $\mathbb{R}^+$with $\Theta \cap \Omega \cap \Psi = \{(0,0,0)\}$. Let $\varrho_1: \Theta \times \Omega \times \Psi \to [0, +\infty)$be defined by $\varrho_1(\theta, \omega, \psi) = |\hat{\theta}| + |\hat{\omega}| + |\hat{\psi}|$. Then $(\Theta, \Omega, \Psi, \varrho_1)$ is a complete stalk space.

(i) θ, ω, ψ $\in$core then clearly

$$\theta = \omega = \psi = (0,0,0)$$

Hence (2.1.1) is satisfied.

(ii) Clearly (2.1.2) is satisfied.

(iii) Suppose $(\theta_1, \omega_1, \psi_1), (\theta_2, \omega_2, \psi_2) \in \Theta \times \Omega \times \Psi$, we show that

$$\varrho_1(\theta_1, \omega_1, \psi_1) \leqslant \varrho_1(\theta_1, \omega_1, \psi_2) + \varrho_1(\theta_2, \omega_2, \psi_2) + \varrho_1(\theta_2, \omega_2, \psi_1).$$

We observe that $\varrho_1(\theta, \omega, \psi) = |\hat{\theta}| + |\hat{\omega}| + |\hat{\psi}|$
From this we have

$$\varrho_1(\theta_1, \omega_1, \psi_1) = |\hat{\theta}_1| + |\hat{\omega}_1| + |\hat{\psi}_1|,$$
$$\varrho_1(\theta_1, \omega_1, \psi_2) = |\hat{\theta}_1| + |\hat{\omega}_1| + |\hat{\psi}_2|$$
$$\varrho_1(\theta_2, \omega_2, \psi_2) = |\hat{\theta}_2| + |\hat{\omega}_2| + |\hat{\psi}_2|$$
$$\text{and}\varrho_1(\theta_2, \omega_2, \psi_1) = |\hat{\theta}_2| + |\hat{\omega}_2| + |\hat{\psi}_1|$$

$$\begin{aligned}\varrho_1(\theta_1, \omega_1, \psi_2) + \varrho_1(\theta_2, \omega_2, \psi_2) + \varrho_1(\theta_2, \omega_2, \psi_1) &= |\hat{\theta}_1| + |\hat{\omega}_1| + |\hat{\psi}_2| + |\hat{\theta}_2| + |\hat{\omega}_2| \\ &\quad + |\hat{\psi}_2| + |\hat{\theta}_2| + |\hat{\omega}_2| + |\hat{\psi}_1| \\ &= \varrho_1(\theta_1, \omega_1, \psi_1) + 2\varrho_1(\theta_2, \omega_2, \psi_2)\end{aligned}$$

$$\begin{aligned}\therefore \varrho_1(\theta_1, \omega_1, \psi_1) &\leqslant \varrho_1(\theta_1, \omega_1, \psi_2) + \varrho_1(\theta_2, \omega_2, \psi_2) + \varrho_1(\theta_2, \omega_2, \psi_1) \\ &= \varrho_1(\theta_1, \omega_1, \psi_1) + 2\varrho_1(\theta_2, \omega_2, \psi_2)\end{aligned}$$

The following can be prove similarly,

$$\varrho_1(\theta_1, \omega_1, \psi_1) \leqslant \varrho_1(\theta_1, \omega_2, \psi_1) + \varrho_1(\theta_2, \omega_1, \psi_2) + \varrho_1(\theta_2, \omega_2, \psi_2),$$
$$\varrho_1(\theta_1, \omega_1, \psi_1) \leqslant \varrho_1(\theta_2, \omega_1, \psi_1) + \varrho_1(\theta_1, \omega_2, \psi_2) + \varrho_1(\theta_2, \omega_2, \psi_2),$$
$$\varrho_1(\theta_2, \omega_2, \psi_2) \leqslant \varrho_1(\theta_1, \omega_1, \psi_2) + \varrho_1(\theta_1, \omega_1, \psi_1) + \varrho_1(\theta_2, \omega_2, \psi_1),$$
$$\varrho_1(\theta_2, \omega_2, \psi_2) \leqslant \varrho_1(\theta_1, \omega_2, \psi_1) + \varrho_1(\theta_1, \omega_1, \psi_1) + \varrho_1(\theta_2, \omega_1, \psi_2)$$

and $\varrho_1(\theta_2, \omega_2, \psi_2) \leqslant \varrho_1(\theta_2, \omega_1, \psi_1) + \varrho_1(\theta_1, \omega_1, \psi_1) + \varrho_1(\theta_1, \omega_2, \psi_2)$.
From this we obtain (2.1.3)
$\therefore$ $(\Theta, \Omega, \Psi, \varrho_1)$is a stalk space.
Suppose $(\theta_n, \omega_n, \psi_n) \in \Theta \times \Omega \times \Psi$and $\varrho_1(\theta_n, \omega_n, \psi_n) \to 0$.
Since $\{(0,0,0)\}$ is the only seed in core, by the Definition-2.3, $(\theta_n, \omega_n, \psi_n) \to (0,0,0)$as $n \to \infty$.
$\therefore$ $(\Theta, \Omega, \Psi, \varrho_1)$is a complete stalk space.

Example 2.7. Let $\Theta = (\mathbb{R}^+, 0,0), \Omega = (0, \mathbb{R}^+, 0)$and $\Psi = (0,0, \mathbb{R}^+)$. Then $\Theta \cap \Omega \cap \Psi = \{(0,0,0)\}$ we write $\theta = (\theta, 0,0), \omega = (0, \omega, 0)$ and $\psi = (0,0, \psi)$. Let$\varrho_2: \Theta \times \Omega \times \Psi \to [0, +\infty)$ be defined by

$$\varrho_2(\theta, \omega, \psi) = \sqrt{\hat{\theta}^2 + \hat{\omega}^2 + \hat{\psi}^2}.$$

Then $(\Theta, \Omega, \Psi, \varrho_2)$ is a complete stalk space.

(i) θ, ω, ψ $\in$core then clearly$\theta = \omega = \psi = (0,0,0)$

Hence (2.1.1) is satisfied.

(ii) Clearly (2.1.2) is satisfied.

(iii) Suppose $(\theta_1, \omega_1, \psi_1), (\theta_2, \omega_2, \psi_2) \in \Theta \times \Omega \times \Psi$, we show that

$$\varrho_2(\theta_1, \omega_1, \psi_1) \leqslant \varrho_2(\theta_1, \omega_1, \psi_2) + \varrho_2(\theta_2, \omega_2, \psi_2) + \varrho_2(\theta_2, \omega_2, \psi_1).$$

We observe that $\varrho_2(\theta, \omega, \psi) = \sqrt{\hat{\theta}^2 + \hat{\omega}^2 + \hat{\psi}^2}$
From this we have

$$\varrho_2(\theta_1, \omega_1, \psi_1) = \sqrt{\hat{\theta}_1{}^2 + \hat{\omega}_1^2 + \hat{\psi}_1^2}$$
$$\varrho_2(\theta_1, \omega_1, \psi_2) = \sqrt{\hat{\theta}_1{}^2 + \hat{\omega}_1^2 + \hat{\psi}_2^2}$$
$$\varrho_2(\theta_2, \omega_2, \psi_2) = \sqrt{\hat{\theta}_2^2 + \hat{\omega}_2^2 + \hat{\psi}_2{}^2}$$
$$\text{and}\varrho_2(\theta_2, \omega_2, \psi_1) = \sqrt{\hat{\theta}_2{}^2 + \hat{\omega}_2^2 + \hat{\psi}_1^2}$$

$$\begin{aligned}\sqrt{\hat{\theta}_1{}^2 + \hat{\omega}_1^2 + \hat{\psi}_1^2} &\leqslant \sqrt{\hat{\theta}_1{}^2 + \hat{\omega}_1^2 + \hat{\psi}_2^2} \\ &\quad + \sqrt{\hat{\theta}_2{}^2 + \hat{\omega}_2^2 + \hat{\psi}_2^2} \\ &\quad + \sqrt{\hat{\theta}_2{}^2 + \hat{\omega}_2^2 + \hat{\psi}_1^2}\end{aligned}$$

$$\therefore \varrho_2(\theta_1, \omega_1, \psi_1) \leqslant \varrho_2(\theta_1, \omega_1, \psi_2) + \varrho_2(\theta_2, \omega_2, \psi_2) + \varrho_2(\theta_2, \omega_2, \psi_1)$$

The following can be prove similarly,

$\varrho_2(\theta_1, \omega_1, \psi_1) \leqslant$
$\varrho_2(\theta_1, \omega_2, \psi_1) + \varrho_2(\theta_2, \omega_1, \psi_2) + \varrho_2(\theta_2, \omega_2, \psi_2)$,
$\varrho_2(\theta_1, \omega_1, \psi_1) \leqslant$
$\varrho_2(\theta_2, \omega_1, \psi_1) + \varrho_2(\theta_1, \omega_2, \psi_2) + \varrho_2(\theta_2, \omega_2, \psi_2)$,
$\varrho_2(\theta_2, \omega_2, \psi_2) \leqslant$
$\varrho_2(\theta_1, \omega_1, \psi_2) + \varrho_2(\theta_1, \omega_1, \psi_1) + \varrho_2(\theta_2, \omega_2, \psi_1)$,
$\varrho_2(\theta_2, \omega_2, \psi_2) \leqslant \varrho_2(\theta_1, \omega_2, \psi_1) + \varrho_2(\theta_1, \omega_1, \psi_1) + \varrho_2(\theta_2, \omega_1, \psi_2)$
and$\varrho_2(\theta_2, \omega_2, \psi_2) \leqslant$
$\varrho_2(\theta_2, \omega_1, \psi_1) + \varrho_2(\theta_1, \omega_1, \psi_1) + \varrho_2(\theta_1, \omega_2, \psi_2)$.
From this we obtain (2.1.3)
$\therefore (\Theta, \Omega, \Psi, \varrho_2)$is a stalk space.
Suppose $(\theta_n, \omega_n, \psi_n) \in \Theta \times \Omega \times \Psi$and
$\varrho_2(\theta_n, \omega_n, \psi_n) \to 0$.
Since {(0,0,0)} is the only seed in core, by the Definition-2.3,
$(\theta_n, \omega_n, \psi_n) \to (0,0,0)asn \to \infty$
$\therefore (\Theta, \Omega, \Psi, \varrho_2)$is a complete stalk space.

Example 2.8. Let $\Theta = (\mathbb{R}^+, 0,0), \Omega = (0, \mathbb{R}^+, 0)$and $\Psi = (0,0, \mathbb{R}^+)$. Then $\Theta \cap \Omega \cap \Psi = \{(0,0,0)\}$ we write $\theta = (\theta, 0,0), \omega = (0, \omega, 0)$ and $\psi = (0,0, \psi)$. Let $\varrho_\infty: \Theta \times \Omega \times \Psi \to [0, +\infty)$ be defined by$\varrho_\infty(\theta, \omega, \psi) = \max\{|\hat{\theta}|, |\hat{\omega}|, |\hat{\psi}|\}$. Then $(\Theta, \Omega, \Psi, \varrho_\infty)$ is a complete stalk space.
(i) $\theta, \omega, \psi \in$core then
clearly$\theta = \omega = \psi = (0,0,0)$
Hence (2.1.1) is satisfied.
(ii) Clearly (2.1.2) is satisfied.
(iii) Suppose $(\theta_1, \omega_1, \psi_1), (\theta_2, \omega_2, \psi_2) \in \Theta \times \Omega \times \Psi$, we show that
$\varrho_\infty(\theta_1, \omega_1, \psi_1) \leqslant$
$\varrho_\infty(\theta_1, \omega_1, \psi_2) + \varrho_\infty(\theta_2, \omega_2, \psi_2)\varrho_\infty(\theta_2, \omega_2, \psi_1)$.
We observe that $\varrho_\infty(\theta, \omega, \psi) = \max\{|\hat{\theta}|, |\hat{\omega}|, |\hat{\psi}|\}$
From this we have
$\varrho_\infty(\theta_1, \omega_1, \psi_1) = \max\{|\hat{\theta}_1|, |\hat{\omega}_1|, |\hat{\psi}_1|\}$,
$\varrho_\infty(\theta_1, \omega_1, \psi_2) = \max\{|\hat{\theta}_1|, |\hat{\omega}_1|, |\hat{\psi}_2|\}$
$\varrho_\infty(\theta_2, \omega_2, \psi_2) = \max\{|\hat{\theta}_2|, |\hat{\omega}_2|, |\hat{\psi}_2|\}$
and$\varrho_\infty(\theta_2, \omega_2, \psi_1) = \max\{|\hat{\theta}_2|, |\hat{\omega}_2|, |\hat{\psi}_1|\}$

$$\begin{aligned}|\hat{\theta}_1| \leqslant \max\{|\hat{\theta}_1|, |\hat{\omega}_1|, |\hat{\psi}_2|\} &\leqslant \max\{|\hat{\theta}_1|, |\hat{\omega}_1|, |\hat{\psi}_2|\} \\ &+ \max\{|\hat{\theta}_2|, |\hat{\omega}_2|, |\hat{\psi}_2|\} \\ &+ \max\{|\hat{\theta}_2|, |\hat{\omega}_2|, |\hat{\psi}_1|\}\end{aligned}$$

$$\begin{aligned}|\hat{\omega}_1| \leqslant \max\{|\hat{\theta}_1|, |\hat{\omega}_1|, |\hat{\psi}_2|\} &\leqslant \max\{|\hat{\theta}_1|, |\hat{\omega}_1|, |\hat{\psi}_2|\} \\ &+ \max\{|\hat{\theta}_2|, |\hat{\omega}_2|, |\hat{\psi}_2|\} \\ &+ \max\{|\hat{\theta}_2|, |\hat{\omega}_2|, |\hat{\psi}_1|\}\end{aligned}$$

$$\begin{aligned}|\hat{\psi}_1| \leqslant \max\{|\hat{\theta}_2|, |\hat{\omega}_2|, |\hat{\psi}_1|\} &\leqslant \max\{|\hat{\theta}_1|, |\hat{\omega}_1|, |\hat{\psi}_2|\} \\ &+ \max\{|\hat{\theta}_2|, |\hat{\omega}_2|, |\hat{\psi}_2|\} \\ &+ \max\{|\hat{\theta}_2|, |\hat{\omega}_2|, |\hat{\psi}_1|\}\end{aligned}$$

$$\therefore \varrho_\infty(\theta_1, \omega_1, \psi_1) \leqslant \varrho_\infty(\theta_1, \omega_1, \psi_2) + \varrho_\infty(\theta_2, \omega_2, \psi_2) + \varrho_\infty(\theta_2, \omega_2, \psi_1)$$

The following can be prove similarly,
$\varrho_\infty(\theta_1, \omega_1, \psi_1) \leqslant \varrho_\infty(\theta_1, \omega_2, \psi_1) + \varrho_\infty(\theta_2, \omega_1, \psi_2) + \varrho_\infty(\theta_2, \omega_2, \psi_2)$
$\varrho_\infty(\theta_1, \omega_1, \psi_1) \leqslant \varrho_\infty(\theta_2, \omega_1, \psi_1) + \varrho_\infty(\theta_1, \omega_2, \psi_2) + \varrho_\infty(\theta_2, \omega_2, \psi_2)$
$\varrho_\infty(\theta_2, \omega_2, \psi_2) \leqslant \varrho_\infty(\theta_1, \omega_1, \psi_2) + \varrho_\infty(\theta_1, \omega_1, \psi_1) + \varrho_\infty(\theta_2, \omega_2, \psi_1)$
$\varrho_\infty(\theta_2, \omega_2, \psi_2) \leqslant \varrho_\infty(\theta_1, \omega_2, \psi_1) + \varrho_\infty(\theta_1, \omega_1, \psi_1) + \varrho_\infty(\theta_2, \omega_1, \psi_2)$
and $\varrho_\infty(\theta_2, \omega_2, \psi_2) \leqslant \varrho_\infty(\theta_2, \omega_1, \psi_1) + \varrho_\infty(\theta_1, \omega_1, \psi_1) + \varrho_\infty(\theta_1, \omega_2, \psi_2)$.
From this we obtain (2.1.3)
$\therefore (\Theta, \Omega, \Psi, \varrho_\infty)$is a stalk space.
Suppose $(\theta_n, \omega_n, \psi_n) \in \Theta \times \Omega \times \Psi$and
$\varrho_\infty(\theta_n, \omega_n, \psi_n) \to 0$.
Since {(0,0,0)} is the only seed in core, by the Definition-2.3,
$(\theta_n, \omega_n, \psi_n) \to (0,0,0)asn \to \infty$
$\therefore (\Theta, \Omega, \Psi, \varrho_\infty)$is a complete stalk space.

Definition 2.9. Let $(\Theta, \Omega, \Psi, \varrho)$ be a stalk space. We say that T is a stalk map on $(\Theta, \Omega, \Psi, \varrho)$ if T is a self map on (Θ, Ω, Ψ) i.e., $T\Theta \subseteq \Theta, T\Omega \subseteq \Omega$and $T\Psi \subseteq \Psi$. If T is a stalk map on $(\Theta, \Omega, \Psi, \varrho)$, we write $T: (\Theta, \Omega, \Psi) \Rrightarrow (\Theta, \Omega, \Psi)$.

If $(\theta, \omega, \psi) \in \Theta \times \Omega \times \Psi$ then we write $T(\theta, \omega, \psi) = (T\theta, T\omega, T\psi)$.

A stalk map T on a stalk space $(\Theta, \Omega, \Psi, \varrho)$ is said to be contraction if$\exists$ $k \in [0,1)$ such that
$\varrho(T\theta, T\omega, T\psi) \leqslant k\varrho(\theta, \omega, \psi)$
for all $\theta, \omega, \psi \in \Theta \times \Omega \times \Psi$.
In this case k is called contraction constant..

Definition 2.10. Suppose T is a stalk map on the stalk space $(\Theta, \Omega, \Psi, \varrho)$. We say that $\ddot{a}$ is a stalk fixed point of$T$ if $\ddot{a} \in$ seed and $T\ddot{a} = \ddot{a}$. In this case we say that $\ddot{a}$ is a Mother of T.

Definition 2.11. A stalk map T on a stalk space $(\Theta, \Omega, \Psi, \varrho)$ is said to satisfy ST-condition if for any given $\epsilon > 0, \exists \delta > 0$ such that

$\epsilon \leqslant \varrho(\theta, \omega, \psi) < \epsilon + \delta$ implies $\varrho(T\theta, T\omega, T\psi) < \epsilon$ **(1)**

whenever $(\Theta, \Omega, \Psi) \in \Theta \times \Omega \times \Psi$.In this case we say that$T$ is a ST-contraction.

We observe that ST-condition is an extension of Meir-keeler condition in metric spaces [7] to stalk spaces.

Definition 2.12.ST-condition for sequences in a stalk space

Suppose $(\Theta, \Omega, \Psi, \varrho)$ is a stalk space and T is a stalk map on $\Theta \times \Omega \times \Psi$. Suppose $\{(\theta_n, \omega_n, \psi_n)\}$ is a sequence in $\Theta \times \Omega \times \Psi$. We say that the sequence satisfies ST-condition for sequences if the following holds,

for any given $\epsilon > 0, \exists\ \delta > 0$ such that

$\epsilon \leqslant \varrho(\theta_m, \omega_n, \psi_l) < \epsilon + \delta$implies

$\varrho(\theta_{m+1}, \omega_{n+1}, \psi_{l+1}) < \epsilon$.

Lemma 2.13. Let $(\Theta, \Omega, \Psi, \varrho)$ be a complete stalk space and $(\theta_n, \omega_n, \psi_n)$ be a sequence in $\Theta \times \Omega \times \Psi$ satisfying the ST-condtion:

For any given $\epsilon > 0 \exists \delta > 0$ such that

$\epsilon \leqslant \varrho(\theta_m, \omega_n, \psi_l) < \epsilon + \delta$ implies

$\varrho(\theta_{m+1}, \omega_{n+1}, \psi_{l+1}) < \epsilon$ **(2)**

Write $\alpha_n = \varrho(\theta_n, \omega_n, \psi_n), \beta_n = \varrho(\theta_n, \omega_n, \psi_{n+1})$,
$\gamma_n = \varrho(\theta_n, \omega_{n+1}, \psi_n), \rho_n = \varrho(\theta_{n+1}, \omega_n, \psi_n)$,
$\sigma_n = \varrho(\theta_{n+1}, \omega_{n+1}, \psi_n)$,
$\Delta_n = \varrho(\theta_{n+1}, \omega_n, \psi_{n+1})$and
$\xi_n = \varrho(\theta_n, \omega_{n+1}, \psi_{n+1})$. Then
$\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\rho_n\}, \{\sigma_n\}, \{\Delta_n\}$ and $\{\xi_n\}$ are monotonic non-increasing and converge to 0.

Proof. (i) Suppose $n = m = l$ and $\varrho(\theta_n, \omega_n, \psi_n) > 0$

Take $\epsilon = \varrho(\theta_n, \omega_n, \psi_n)$. Then by the equation $\varrho(\theta_{n+1}, \omega_{n+1}, \psi_{n+1}) < \epsilon$ so that $\alpha_{n+1} < \alpha_n$(say)

$\therefore \varrho(\theta_n, \omega_n, \psi_n) \downarrow \eta$, where $\eta \geqslant 0$. Then

$\varrho(\theta_n, \omega_n, \psi_n) \geqslant \eta$ **(3)**

If $\eta > 0\ \exists\ \delta > 0$ such that *(2)* holds.

Hence $\exists\ n_0 \in \mathbb{N}$ such that $n \geqslant n_0$

$\Rightarrow \varrho(\theta_n, \omega_n, \psi_n) < \eta + \delta$

$\therefore \varrho(\theta_{n+1}, \omega_{n+1}, \psi_{n+1}) < \eta$for$n \geqslant n_0$ which is a contradiction to *(3)*

Hence $\eta = 0$

$\therefore \varrho(\theta_n, \omega_n, \psi_n) \downarrow 0$

$\therefore \alpha_n \to 0$as $n \to \infty$.

In a similar way we can show that $\{\beta_n\}, \{\gamma_n\}, \{\rho_n\}, \{\sigma_n\}, \{\Delta_n\}and\{\xi_n\}$ decrease to 0.

Lemma 2.14. Every contraction in stalk space is a ST-contraction.

Proof. Suppose T is a contraction so that $\varrho(T\theta, T\omega, T\psi) \leqslant k\varrho(\theta, \omega, \psi)$, where $k \in [0,1)$.

Suppose $\epsilon > 0$. Take $\delta = \frac{(1-k)}{k}\epsilon$.

Suppose $\epsilon \leqslant \varrho(\theta, \omega, \psi) < \epsilon + \delta$.

Then $\varrho(T\theta, T\omega, T\psi) \leqslant k\varrho(\theta, \omega, \psi) < k(\epsilon + \delta)$

$$= k\epsilon + k\left(\frac{1-k}{k}\right)\epsilon = \epsilon$$

$\therefore \varrho(T\theta, T\omega, T\psi) < \epsilon$

$\therefore$for any given $\epsilon > 0\ \exists\ \delta > 0$ such that

$\epsilon \leqslant \varrho(\theta, \omega, \psi) < \epsilon$implies $\varrho(T\theta, T\omega, T\psi) < \epsilon$.

Theorem 2.15. Suppose $(\Theta, \Omega, \Psi, \varrho)$ is a complete stalk space and T is a stalk map on $(\Theta, \Omega, \Psi, \varrho)$ satisfying the ST-condition:

For any given $\epsilon > 0$ there exists $\delta > 0$ such that

$\epsilon \leqslant \varrho(\theta, \omega, \psi) < \epsilon + \delta$ implies $\varrho(T\theta, T\omega, T\psi) < \epsilon$**(4)**

Then T has only one seed, $\ddot{a} \in$ core $\Theta \cap \Omega \cap \Psi$ which is a mother, i.e., $T\ddot{a} = \ddot{a}$.

Proof. Let $(\Theta, \Omega, \Psi, \varrho)$ be a complete stalk space and $T: (\Theta, \Omega, \Psi) \Rrightarrow (\Theta, \Omega, \Psi)$ be a stalk map.

For any element $(\theta, \omega, \psi) \in \Theta \times \Omega \times \Psi$. We write $T(\theta, \omega, \psi)$as$(T\theta, T\omega, T\psi)$.

Now let $(\theta_0, \omega_0, \psi_0)$ be an element in $\Theta \times \Omega \times \Psi$.

Define the sequence $\{(\theta_n, \omega_n, \psi_n)\}$ in $\Theta \times \Omega \times \Psi$ as follows

$(\theta_1, \omega_1, \psi_1) = T(\theta_0, \omega_0, \psi_0)$

$(\theta_2, \omega_2, \psi_2) = T^2(\theta_0, \omega_0, \psi_0)$

$= T\big(T(\theta_0, \omega_0, \psi_0)\big) = T(\theta_1, \omega_1, \psi_1)$

$(\theta_3, \omega_3, \psi_3) = T(\theta_2, \omega_2, \psi_2), \dots, (\theta_{n+1}, \omega_{n+1}, \psi_{n+1})$

$= (T\theta_n, T\omega_n, T\psi_n)\ \forall\ n = 0,1,2, \dots$

$\therefore \{(\theta_n, \omega_n, \psi_n)\}$is a sequence in $\Theta \times \Omega \times \Psi$.

Write $\alpha_n = \varrho(\theta_n, \omega_n, \psi_n)$.

Now we show that $\{\alpha_n\}$ is monotonic non-increasing sequence.

we may suppose that $\varrho(\theta_n, \omega_n, \psi_n) > 0$

we observe that $\alpha_n = 0$ implies $\alpha_{n+1} = 0$

Take $\epsilon = \varrho(\theta_n, \omega_n, \psi_n)$. Then by equation-(4),
$\varrho(T\theta_n, T\omega_n, T\psi_n) < \epsilon$implies $\alpha_{n+1} < \alpha_n$
Suppose $\varrho(\theta_n, \omega_n, \psi_n) \downarrow \eta$, where $\eta \geqslant 0$.
$\therefore \varrho(\theta_n, \omega_n, \psi_n) \geqslant \eta$ **(5)**
Suppose $\eta > 0$. Then $\exists\ \delta > 0$ such that equation-(4) holds.
$\exists\ n_0 \in \mathbb{N}$such that $n \geqslant n_0$ implies
$\varrho(\theta_n, \omega_n, \psi_n) < \eta + \delta$ for $n \geqslant n_0$
$\therefore \varrho(\theta_{n+1}, \omega_{n+1}, \psi_{n+1}) < \eta$for $n \geqslant n_0$.
This contradicts equation-(5)
$\therefore \eta = 0$
Hence $\varrho(\theta_n, \omega_n, \psi_n) \downarrow 0$
$\therefore \exists\ \ddot{a}$ ∈seed i.e., $(\ddot{a}, \ddot{a}, \ddot{a}) \in$ core $\Theta \cap \Omega \cap \Psi$ such that $\varrho(\theta_n, \ddot{a}, \ddot{a}) \to 0, \varrho(\ddot{a}, \omega_n, \ddot{a}) \to 0$,
$\varrho(\ddot{a}, \ddot{a}, \psi_n) \to 0, \varrho(\theta_n, \omega_n, \ddot{a}) \to 0$,
$\varrho(\theta_n, \ddot{a}, \psi_n) \to 0and\varrho(\ddot{a}, \omega_n, \psi_n) \to 0$.
Claim:$(T\ddot{a}, T\ddot{a}, T\ddot{a}) = (\ddot{a}, \ddot{a}, \ddot{a})$
First we show that $\varrho(\theta_{n+1}, T\ddot{a}, T\ddot{a}) \to 0$
Now $\varrho(\theta_n, \ddot{a}, \ddot{a}) = 0 \Rightarrow \theta_n = \ddot{a} = \ddot{a}$
$\Rightarrow T\theta_n = T\ddot{a}$
$\Rightarrow \varrho(\theta_{n+1}, T\ddot{a}, T\ddot{a}) = 0$.
Let $\epsilon > 0$. Then $\exists\ n_0 \in \mathbb{N}$
such that $\varrho(\theta_n, \ddot{a}, \ddot{a}) < \epsilon \quad \forall n \geqslant n_0$
We may suppose that $0 < \varrho(\theta_n, \ddot{a}, \ddot{a}) = \epsilon$
Then by equation- (4),
$\varrho(\theta_{n+1}, T\ddot{a}, T\ddot{a}) = \varrho(T\theta_n, T\ddot{a}, T\ddot{a})$
$< \varrho(\theta_n, \ddot{a}, \ddot{a}) \to 0$as $n \to \infty$.
$\therefore \varrho(\theta_{n+1}, T\ddot{a}, T\ddot{a}) \to 0asn \to \infty$.
Similarly we can show that,
$\varrho(T\ddot{a}, \omega_{n+1}, T\ddot{a}) \to 0$as $n \to \infty$,
$\varrho(T\ddot{a}, T\ddot{a}, \psi_{n+1}) \to 0asn \to \infty$
$\varrho(\theta_{n+1}, \omega_{n+1}, T\ddot{a}) \to 0$as $n \to \infty$,
$\varrho(\theta_{n+1}, T\ddot{a}, \psi_{n+1}) \to 0asn \to \infty$
and$\varrho(T\ddot{a}, \omega_{n+1}, \psi_{n+1}) \to 0$ as $n \to \infty$.
$\therefore (\theta_n, \omega_n, \psi_n) \to (T\ddot{a}, T\ddot{a}, T\ddot{a})$by Lemma-2.4,
$\therefore T\ddot{a} = \ddot{a}$
$\therefore \ddot{a}$is a mother of T.
Let $(\ddot{b}, \ddot{b}, \ddot{b}) \in$ core $\Theta \cap \Omega \cap \Psi$ be another mother of T i.e., $T\ddot{b} = \ddot{b}$.
Suppose $\ddot{a} \neq \ddot{b}$. Then $\varrho(\ddot{b}, \ddot{b}, \ddot{a}) > 0$
Let $\epsilon = \varrho(\ddot{b}, \ddot{b}, \ddot{a})$. Then from equation-(4),
$\varrho(T\ddot{b}, T\ddot{b}, T\ddot{a}) < \epsilon$
$\varrho(\ddot{b}, \ddot{b}, \ddot{a}) < \epsilon = \varrho(\ddot{b}, \ddot{b}, a)$
$\Rightarrow \varrho(\ddot{b}, \ddot{b}, \ddot{a}) < \varrho(\ddot{b}, \ddot{b}, \ddot{a})$, which is a contradiction
$\therefore \ddot{a} = \ddot{b}$
Thus T has only one mother.
Now we give an Example in support of Theorem-2.15.

Example 2.16. Write $\Theta = (\mathbb{R}^+, 0,0)$, $\Omega = (0, \mathbb{R}^+, 0)and\Psi = (0,0, \mathbb{R}^+)$in this case we say that Θ, Ω and Ψ are independent copies of $\mathbb{R}^+$with $\Theta \cap \Omega \cap \Psi = \{(0,0,0)\}$. Let $\varrho_2 : \Theta \times \Omega \times \Psi \to [0, +\infty)$be defined by
$\varrho_2(\theta, \omega, \psi) =$
$\| \theta - \psi \|_2 + \| \omega - \psi \|_2 + \| \theta + \omega - 2\psi \|_2$,
$\forall\ \theta, \omega, \psi \in \mathbb{R}$. Then $(\Theta, \Omega, \Psi, \varrho_2)$ is a complete stalk space. Let $T(\Theta, \Omega, \Psi) \Rrightarrow (\Theta, \Omega, \Psi)$ be a stalk map defined by $T(\theta, 0,0) = \left(\frac{\theta}{2}, 0,0\right)$,
$T(0, \omega, 0) = \left(0, \frac{\omega}{2}, 0\right)andT(0,0, \psi) = \left(0,0, \frac{\psi}{2}\right)$.
Thus $T\theta = \frac{\theta}{2}; T\omega = \frac{\omega}{2}; T\psi = \frac{\psi}{2}$.
and$T^{n+1}\theta = \frac{\theta}{2^{n+1}}, \forall\ n = 0,1,2, \ldots$
Similarly, $T^{n+1}\omega = \frac{\omega}{2^{n+1}}$ and$T^{n+1}\psi = \frac{\psi}{2^{n+1}}$
for$\omega \in \Omega$ and $\psi \in \Psi$.
Now the ST-condition is
For any given $\epsilon > 0$take $\delta = \frac{\epsilon}{2}$.
Then $\varrho_2(\theta, \omega, \psi) =$
$$\sqrt{\hat{\theta}^2 + \hat{\psi}^2} + \sqrt{\hat{\omega}^2 + \hat{\psi}^2} + \sqrt{\hat{\theta}^2 + \hat{\omega}^2 + 4\hat{\psi}^2}$$
and$\varrho_2(T\theta, T\omega, T\psi) = \sqrt{\frac{1}{4}\left(\hat{\theta}^2 + \hat{\psi}^2\right)} +$
$$\sqrt{\frac{1}{4}\left(\hat{\omega}^2 + \hat{\psi}^2\right)} + \sqrt{\frac{1}{4}\left(\hat{\theta}^2 + \hat{\omega}^2 + 4\hat{\psi}^2\right)}$$
$$= \frac{1}{2}\left(\sqrt{\hat{\theta}^2 + \hat{\psi}^2} + \sqrt{\hat{\omega}^2 + \hat{\psi}^2} + \sqrt{\hat{\theta}^2 + \hat{\omega}^2 + 4\hat{\psi}^2}\right)$$
$$= \frac{1}{2}\varrho_2(\theta, \omega, \psi)$$
Now, $\epsilon \leqslant \varrho_2(\theta, \omega, \psi) < \epsilon + \delta = \epsilon + \frac{\epsilon}{2} = \frac{3\epsilon}{2}$
$\therefore \frac{\epsilon}{2} \leqslant \frac{1}{2}\varrho_2(\theta, \omega, \psi) < \frac{3\epsilon}{4} < \epsilon$
$\therefore \varrho_2(T\theta, T\omega, T\psi) < \epsilon$
Thus condtion-(4) of Theorem-2.15, holds.
$\therefore$by Definition-2.10, T has only one seed which is a mother.
Clearly, $(0,0,0)$ is the only mother of T.
Now we obtain contraction principle in stalk spaces as a Corollary to Theorem-2.15.

Corollary 2.17.Contraction principle in stalk spaces Let $(\Theta, \Omega, \Psi, \varrho)$ be a complete stalk space and $T: (\Theta, \Omega, \Psi) \Rrightarrow (\Theta, \Omega, \Psi)$ be a contraction i.e., $\exists\ k \in [0,1)$ such that

$$\varrho(T\theta, T\omega, T\psi) \leqslant k\varrho(\theta, \omega, \psi)$$

for all $(\Theta, \Omega, \Psi) \in \Theta \times \Omega \times \Psi$. Then T has a only mother.

Proof. Follows from Lemma-2.14 and Theorem-2.15.

Example 2.18. Write $\Theta = (\mathbb{R}^+, 0,0), \Omega = (0, \mathbb{R}^+, 0)$ and $\Psi = (0,0, \mathbb{R}^+)$in this case we say that

Θ, ΩandΨ are independent copies of $\mathbb{R}^+$with $\Theta \cap \Omega \cap \Psi = \{(0,0,0)\}$. Define $\varrho_0: \Theta \times \Omega \times \Psi \to [0, +\infty)$by

$$\varrho_0(\theta, \omega, \psi) = \| \theta - \psi \|_1 + \| \omega - \psi \|_1 + \| \theta + \omega - 2\psi \|_1$$

Write $\theta = (\theta, 0,0), \omega = (0, \omega, 0)$ and $\psi = (0,0, \psi) \forall \theta, \omega, \psi \in \mathbb{R}$. Let $T: (\Theta, \Omega, \Psi) \Rrightarrow (\Theta, \Omega, \Psi)$be defined by $T(\theta, 0,0) = \left(\frac{\theta}{2}, 0,0\right), T(0, \omega, 0) = \left(0, \frac{\omega}{2}, 0\right)$ and $T(0,0, \psi) = \left(0,0, \frac{\psi}{2}\right)$. Thus T is a contraction with contraction constant $\frac{1}{2}$.

Clearly by Example- (2.5), $(\Theta, \Omega, \Psi, \varrho_0)$ is a complete stalk space.

Let $\theta_0 = (1,0,0), \omega_0 = (0,1,0)$and $\psi_0 = (0,0,1)$. Write

$$\theta_1 = T\theta_0 = \left(\frac{1}{2}, 0,0\right), \omega_1 = T\omega_0 = \left(0, \frac{1}{2}, 0\right)$$

and $\psi_1 = T\psi_0 = \left(0,0, \frac{1}{2}\right)$.

Inductively, $\theta_{n+1} = T\theta_n, \omega_{n+1} = T\omega_n$and $\psi_{n+1} = T\psi_n$.

So that

$$\begin{aligned}\varrho_0(\theta_{n+1}, \omega_{n+1}, \psi_{n+1}) &= \varrho_0(T\theta_n, T\omega_n, T\psi_n) \\ &= \varrho_0\left(\frac{\theta_n}{2}, \frac{\omega_n}{2}, \frac{\psi_n}{2}\right) \\ &= \frac{1}{2}\varrho_0(\theta_0, \omega_0, \psi_0) \\ &\leqslant \frac{1}{2^{n+1}}\varrho_0(\theta_0, \omega_0, \psi_0)\end{aligned}$$

$\therefore \varrho_0(\theta_n, \omega_n, \psi_n) \to 0$ as $n \to \infty$.

Now $(0,0,0) \in \Theta \cap \Omega \cap \Psi$ such that $\varrho_0(\theta_n, 0,0) \to 0, \varrho_0(0, \omega_n, 0) \to 0$, $\varrho_0(0,0, \psi_n) \to 0$, $\varrho_0(\theta_n, \omega_n, 0) \to 0$, $\varrho_0(\theta_n, 0, \psi_n) \to 0and\varrho_0(0, \omega_n, \psi_n) \to 0$.

$\therefore (\theta_n, \omega_n, \psi_n) \to (T0, T0, T0)$

$\therefore (T0, T0, T0) = (0,0,0)$

$\therefore$ 0is the only mother of T.

The following Example supports Theorem-2.15, being a ST-contraction.

Example 2.19. Write $\Theta = (\mathbb{R}^+, 0,0), \Omega = (0, \mathbb{R}^+, 0)$and $\Psi = (0,0, \mathbb{R}^+)$. Let $\varrho_0: \Theta \times \Omega \times \Psi \to [0, +\infty)$ be defined by

$$\varrho_0(\theta, \omega, \psi) = \| \theta - \psi \|_1 + \| \omega - \psi \|_1 + \| \theta + \omega - 2\psi \|_1.$$

Then $(\Theta, \Omega, \Psi, \varrho_0)$ is a complete stalk space Example-(2.5). Define the stalk map $T: (\Theta, \Omega, \Psi) \Rrightarrow (\Theta, \Omega, \Psi)$ by $T(\theta, 0,0) = \left(\frac{\theta}{2}, 0,0\right), T(0, \omega, 0) = \left(0, \frac{\omega}{3}, 0\right)$and $T(0,0, \psi) = \left(0,0, \frac{\psi}{5}\right)$. Then T is a ST-contraction.

Let $\epsilon > 0$ and take $\delta = \frac{\epsilon}{7}$

We have $\epsilon \leqslant \varrho_0(\theta, \omega, \psi) < \epsilon + \frac{\epsilon}{7} = \frac{8\epsilon}{7}$

$$\varrho_0(\theta, \omega, \psi) = 2\hat{\theta} + 2\hat{\omega} + 4\hat{\psi} < \frac{8\epsilon}{7} \qquad \textbf{(6)}$$

$$\Rightarrow \hat{\theta} + \hat{\omega} + 2\hat{\psi} < \frac{4\epsilon}{7} \qquad \textbf{(7)}$$

Now $\varrho_0(T\theta, T\omega, T\psi) = \left\|\frac{\hat{\theta}}{2} - \frac{\hat{\psi}}{5}\right\|_1 + \left\|\frac{\hat{\omega}}{3} - \frac{\hat{\psi}}{5}\right\|_1 + \left\|\frac{\hat{\theta}}{2} + \frac{\hat{\omega}}{3} - 2\frac{\hat{\psi}}{5}\right\|_1$

$$\begin{aligned}&= \left|\frac{\hat{\theta}}{2}\right| + \left|\frac{\hat{\psi}}{5}\right| + \left|\frac{\hat{\omega}}{3}\right| + \left|\frac{\hat{\psi}}{5}\right| + \left|\frac{\hat{\theta}}{2}\right| + \left|\frac{\hat{\omega}}{3}\right| + 2\left|\frac{\hat{\psi}}{5}\right| \\ &= \hat{\theta} + 2\left(\frac{\hat{\omega}}{3}\right) + 4\left(\frac{\hat{\psi}}{5}\right) \\ &\leqslant \hat{\theta} + \hat{\omega} + 2\hat{\psi} \\ &< \frac{4\epsilon}{7} \text{ by equation-(7) } < \epsilon.\end{aligned}$$

$\therefore \varrho_0(T\theta, T\omega, T\psi) < \epsilon$

$\therefore T$is a ST-contraction.

III. Stalk continuity, commuting Stalk maps and weakly compatible Stalk maps

Definition 3.1.Stalk continuity:

Suppose $(\Theta, \Omega, \Psi, \varrho)$ is a stalk space and let $S: (\Theta, \Omega, \Psi) \Rrightarrow (\Theta, \Omega, \Psi)$be a stalk map. If a sequence, $\left(\ddot{a}_n, \ddot{b}_n, \ddot{c}_n\right) \to (\ddot{a}, \ddot{a}, \ddot{a}) \in$ core implies $S\left(\ddot{a}_n, \ddot{b}_n, \ddot{c}_n\right) \to S(\ddot{a}, \ddot{a}, \ddot{a})$ then we say that S is continuous.

Definition 3.2. Suppose $(\Theta,\Omega,\Psi,\varrho)$ is a stalk space and $S,T:(\Theta,\Omega,\Psi)\Rrightarrow(\Theta,\Omega,\Psi)$ are two stalk maps. Suppose the pair (S,T) satisfies the following conditions:
$\forall(\theta,\omega,\psi)\in\Theta\times\Omega\times\Psi$
(3.2.1.) for any given $\epsilon>0\exists\delta>0$ such that $\epsilon\leqslant\varrho(S\theta,S\omega,S\psi)<\epsilon+\delta$ implies $\varrho(T\theta,T\omega,T\psi)<\epsilon$ and
(3.2.2.) $S\theta=S\omega=S\psi$ implies $T\theta=T\omega=T\psi$.
Then we say that T is a ST-contraction with respect to S (or) simply (S,T) is a ST-contraction pair.

Note: If S is the identity map, we get Definition-2.11.
Now we have the following Theorem for a ST-contraction pair of stalk maps.

Theorem 3.3. Let $(\Theta,\Omega,\Psi,\varrho)$ be a complete stalk space and let $S,T:(\Theta,\Omega,\Psi)\Rrightarrow(\Theta,\Omega,\Psi)$ be two stalk maps such that T is a ST-contraction with respect to S. Then T is continuous if S is continuous.
Proof. Suppose T is a ST-contraction with respect to S.
For any given $\epsilon>0$ there exists $\delta>0$ such that
$\epsilon\leqslant\varrho(S\theta,S\omega,S\psi)<\epsilon+\delta$
implies $\varrho(T\theta,T\omega,T\psi)<\epsilon$ **(8)**
and $S\theta=S\omega=S\psi$ implies $T\theta=T\omega=T\psi$ **(9)**
$\forall\,(\theta,\omega,\psi)\in\Theta\times\Omega\times\Psi$.
Suppose S is continuous.
Suppose $(\theta_n,\omega_n,\psi_n)$ is a sequence in $\Theta\times\Omega\times\Psi$ such that $\varrho(S\theta_n,S\omega_n,S\psi_n)\to 0$.
Then by equation-(8)
$$\varrho(T\theta_n,T\omega_n,T\psi_n)\leqslant\varrho(S\theta_n,S\omega_n,S\psi_n)\to 0$$
$\therefore\varrho(T\theta_n,T\omega_n,T\psi_n)\to 0$
Further, if $(\theta_n,\omega_n,\psi_n)\to(\ddot{a},\ddot{a},\ddot{a})$ and S is continuous then $(S\theta_n,S\omega_n,S\psi_n)\to(S\ddot{a},S\ddot{a},S\ddot{a})$
Now, $\varrho(T\theta_n,T\ddot{a},T\ddot{a})\leqslant\varrho(S\theta_n,S\ddot{a},S\ddot{a})\to 0$
$\varrho(T\ddot{a},T\omega_n,T\ddot{a})\leqslant\varrho(S\ddot{a},S\omega_n,S\ddot{a})\to 0$
$\varrho(T\ddot{a},T\ddot{a},T\psi_n)\leqslant\varrho(S\ddot{a},S\ddot{a},S\psi_n)\to 0$
$\varrho(T\theta_n,T\omega_n,T\ddot{a})\leqslant\varrho(S\theta_n,S\omega_n,S\ddot{a})\to 0$
$\varrho(T\theta_n,T\ddot{a},T\psi_n)\leqslant\varrho(S\theta a_n,S\ddot{a},S\psi_n)\to 0$
$\varrho(T\ddot{a},T\omega_n,T\psi_n)\leqslant\varrho(S\ddot{a},S\omega_n,S\psi_n)\to 0$
$\therefore(T\theta_n,T\omega_n,T\psi_n)\to(T\ddot{a},T\ddot{a},T\ddot{a})$
$\therefore T$ is continuous.

Theorem 3.4. Let $(\Theta,\Omega,\Psi,\varrho)$ be a complete stalk space and let $S,T:(\Theta,\Omega,\Psi)\Rrightarrow(\Theta,\Omega,\Psi)$ be two stalk maps which satisfy the following conditions:
(3.4.1). S and T are commuting mappings (ie., $ST=TS$)
(3.4.2). S is continuous
(3.4.3). $T\Theta\subseteq S\Theta, T\Omega\subseteq S\Omega$ and $T\Psi\subseteq S\Psi$
(3.4.4). T is a ST-contraction with respect to S.
Then the maps S and T have only one common mother.
Proof. Let $\theta_0\in\Theta,\omega_0\in\Omega$ and $\psi_0\in\Psi$.
Now choose $\theta_1\in\Theta$ such that $T\theta_0=S\theta_1=u_1,\omega_1\in\Omega$ such that $T\omega_0=S\omega_1=v_1$ and $\psi_1\in\psi$ such that $T\psi_0=S\psi_1=w_1$. This can be done by condition-(3.4.3).
In general we can choose $(\theta_n,\omega_n,\psi_n)\in\Theta\times\Omega\times\Psi$ such that $T\theta_{n-1}=S\theta_n=u_n,T\omega_{n-1}=S\omega_n=v_n$ and $T\psi_{n-1}=S\psi_n=w_n,\forall n\in\mathbb{N}$.
$$\text{Hence}(T\theta_{n-1},T\omega_{n-1},T\psi_{n-1})=(S\theta_n,S\omega_n,S\psi_n)=(u_n,v_n,w_n)=P_n(\text{say})$$
Claim: $\varrho(P_n)\downarrow 0$.
Take $\epsilon=\varrho(P_n)$. Then by condition-(3.4.4), $\varrho(u_{n+1},v_{n+1},w_{n+1})<\epsilon$ implies $\varrho(P_{n+1})<\varrho(P_n)$
$\therefore\varrho(P_n)\downarrow\eta$, where $\eta\geqslant 0$. Then
$\varrho(P_n)\geqslant\eta\ \forall\ n\in\mathbb{N}$ **(10)**
If $\eta>0$ there exists $\delta>0$ such that condition-(3.4.4) holds
i.e., $\eta\leqslant\varrho(S\theta_n,S\omega_n,S\psi_n)<\eta+\delta$ for large n
$\Rightarrow\varrho(T\theta_n,T\omega_n,T\psi_n)<\eta$
$\Rightarrow\varrho(P_{n+1})<\eta$
which is a contradiction to (10).
Hence $\eta=0$
$\therefore\varrho(P_n)\downarrow 0$
$\exists\ddot{q}\in$seed i.e., $\ddot{Q}=(\ddot{q},\ddot{q},\ddot{q})\in$ core $\Theta\cap\Omega\cap\Psi$ such that $P_n\to\ddot{Q}\Rightarrow SP_n\to S\ddot{Q}$
$$\Rightarrow TP_n\to T\ddot{Q}$$
(Since T is continuous by Theorem-3.3)
$TP_n=S(P_{n+1})\to S\ddot{Q}$
$\therefore S\ddot{Q}=T\ddot{Q}$
$\Rightarrow S(S\ddot{Q})=S(T\ddot{Q})$
Claim: $\ddot{Q}$ is a mother of S and T.
Suppose $S\ddot{Q}\neq\ddot{Q}$.
Then $\varrho(S(S\ddot{Q}),S\ddot{Q},S\ddot{Q})>0$
$\therefore\varrho(T(S\ddot{Q}),T\ddot{Q},T\ddot{Q})<\varrho(S(S\ddot{Q}),S\ddot{Q},S\ddot{Q})$
$\therefore\varrho\big(T(S\ddot{Q}),T\ddot{Q},T\ddot{Q}\big)<\varrho\big(S(T\ddot{Q}),T\ddot{Q},T\ddot{Q}\big)$

($\because$by condition-3.4.1)
$\therefore \varrho(T(S\ddot{Q}), T\ddot{Q}, T\ddot{Q}) < \varrho(T(S\ddot{Q}), T\ddot{Q}, T\ddot{Q})$, Which is a contradiction$\therefore \varrho(S(S\ddot{Q}), S\ddot{Q}, S\ddot{Q}) = 0$
$\therefore S\ddot{Q} = \ddot{Q} \Rightarrow T\ddot{Q} = \ddot{Q}$
$\therefore \ddot{Q}$is a common mother of$S$ and T.
Let $\ddot{Q}, \ddot{R}$ be mothers of $S$and T.
Suppose $\ddot{Q} \neq \ddot{R}$.
Then $\varrho(\ddot{Q}, \ddot{Q}, \ddot{R}) = \varrho(T\ddot{Q}, T\ddot{Q}, T\ddot{R})$
$< \varrho(S\ddot{Q}, S\ddot{Q}, S\ddot{R}) = \varrho(\ddot{Q}, \ddot{Q}, \ddot{R})$
$\therefore \varrho(\ddot{Q}, \ddot{Q}, \ddot{R}) < \varrho(\ddot{Q}, \ddot{Q}, \ddot{R})$which is a contradiction
$\therefore \varrho(\ddot{Q}, \ddot{Q}, \ddot{R}) = 0$
$\therefore \ddot{Q} = \ddot{R}$
Thus S and T have only one common mother.

Definition 3.5. Weakly compatible stalk maps: Suppose $(\Theta, \Omega, \Psi, \varrho)$ is a complete stalk space and $S, T: (\Theta, \Omega, \Psi) \Rrightarrow (\Theta, \Omega, \Psi)$ are two stalk maps. If $(S\ddot{a}, S\ddot{b}, S\ddot{c}) = (T\ddot{a}, T\ddot{b}, T\ddot{c})$ implies $(TS\ddot{a}, TS\ddot{b}, TS\ddot{c}) = (ST\ddot{a}, ST\ddot{b}, ST\ddot{c})$ then we say that S and T are weakly compatible stalk maps.

Theorem 3.6. Let $(\Theta, \Omega, \Psi, \varrho)$ be a complete stalk space and let $S, T: (\Theta, \Omega, \Psi) \Rrightarrow (\Theta, \Omega, \Psi)$ be two stalk maps which satisfy the following conditions:
(3.6.1). S and T are weakly compatible.
(3.6.2). $S(\Theta \cup \Omega \cup \Psi)$ is complete.
(3.6.3). S is one-one.
(3.6.4). $T\Theta \subseteq S\Theta, T\Omega \subseteq S\Omega$and $T\Psi \subseteq S\Psi$.
(3.6.5). T is a ST-contraction with respect to S.
Then S and T have one common mother.

Proof. Let $\theta_0 \in \Theta, \omega_0 \in \Omega$and $\psi_0 \in \Psi$.
Now choose $\theta_1 \in \Theta$ such that $T\theta_0 = S\theta_1 = u_1, \omega_1 \in \Omega$ such that $T\omega_0 = S\omega_1 = v_1and\psi_1 \in \Psi$ such that $T\psi_0 = S\psi_1 = w_1$. This can be done by condition-(3.6.4).
In general we can choose $(\theta_n, \omega_n, \psi_n) \in \Theta \times \Omega \times \Psi$ such that $T\theta_{n-1} = S\theta_n = u_n, T\omega_{n-1} = S\omega_n = v_n$and $T\psi_{n-1} = S\psi_n = w_n, \forall n \in \mathbb{N}$.
Hence $(T\theta_{n-1}, T\omega_{n-1}, T\psi_{n-1}) = (S\theta_n, S\omega_n, S\psi_n)$
Claim:$\varrho(S\theta_n, S\omega_n, S\psi_n) \downarrow 0$
Take $\epsilon = \varrho(S\theta_n, S\omega_n, S\psi_n)$
Then by condition- (3.6.5),$\varrho(T\theta_n, T\omega_n, T\psi_n) < \epsilon = \varrho(S\theta_n, S\omega_n, S\psi_n)$
$\therefore \varrho(S\theta_n, S\omega_n, S\psi_n) \downarrow \eta$, where $\eta \geqslant 0$
$\therefore \varrho(S\theta_n, S\omega_n, S\psi_n) \geqslant \eta \ \forall\ n \in \mathbb{N}$ **(11)**
If $\eta > 0$ there exists $\delta > 0$ such that condition-(3.6.5) holds
i.e., $\eta \leqslant \varrho(S\theta_n, S\omega_n, S\psi_n) < \eta + \delta$ for large n
$\Rightarrow \varrho(T\theta_n, T\omega_n, T\psi_n) < \eta$
$\Rightarrow \varrho(S\theta_{n+1}, S\omega_{n+1}, S\psi_{n+1}) < \eta$for large n
which contradicts (11)
Hence $\eta = 0$
$\therefore \varrho(S\theta_n, S\omega_n, S\psi_n) \downarrow 0$
By condition-(3.6.2), $\exists\ \ddot{q} \in$ seed
i.e., $\ddot{Q} = (\ddot{q}, \ddot{q}, \ddot{q}) \in$ core $\Theta \cap \Omega \cap \Psi$ such that
$(S\theta_n, S\omega_n, S\psi_n) \to S\ddot{Q} = S(\ddot{q}, \ddot{q}, \ddot{q})$
$= (S\ddot{q}, S\ddot{q}, S\ddot{q})$
$(T\theta_n, T\omega_n, T\psi_n) = (S\theta_{n+1}, S\omega_{n+1}, S\psi_{n+1}) \to SQ$
Now by using condition (3.6.5),
$\varrho(S\theta_{n+1}, S\omega_{n+1}, S\ddot{Q}) \to$ 0implies$\varrho(T\theta_n, T\omega_n, T\ddot{Q}) \to 0$
$\varrho(S\theta_{n+1}, S\ddot{Q}, S\psi_{n+1}) \to$ 0implies$\varrho(T\theta_n, T\ddot{Q}, T\psi_n) \to 0$
$\varrho(S\ddot{Q}, S\omega_{n+1}, S\psi_{n+1}) \to$ 0implies$\varrho(T\ddot{Q}, T\omega_n, T\psi_n) \to 0$
$\varrho(S\theta_{n+1}, S\ddot{Q}, S\ddot{Q}) \to$ 0implies $\varrho(T\theta_n, T\ddot{Q}, T\ddot{Q}) \to 0$
$\varrho(S\ddot{Q}, S\omega_{n+1}, S\ddot{Q}) \to$ 0implies$\varrho(T\ddot{Q}, T\omega_n, T\ddot{Q}) \to 0$
$\varrho(S\ddot{Q}, S\ddot{Q}, S\psi_{n+1}) \to$ 0implies$\varrho(T\ddot{Q}, T\ddot{Q}, T\psi_n) \to 0$
$\varrho(T\theta_n, T\omega_n, T\psi_n) \to TQ$
$\therefore S\ddot{Q} = T\ddot{Q}$
$\Rightarrow T(S\ddot{Q}) = S(T\ddot{Q})$($\because$by condition- (3.6.1)).
If $S\ddot{Q} \neq \ddot{Q} \cdot \varrho(S(S\ddot{Q}), S\ddot{Q}, S\ddot{Q}) > 0$
$\Rightarrow \varrho(T(S\ddot{Q}), T\ddot{Q}, T\ddot{Q}) < \varrho(S(S\ddot{Q}), S\ddot{Q}, S\ddot{Q})$
$= \varrho(S(T\ddot{Q}), S\ddot{Q}, S\ddot{Q})$
$\Rightarrow \varrho(T(S\ddot{Q}), T\ddot{Q}, T\ddot{Q}) < \varrho(T(S\ddot{Q}), T\ddot{Q}, T\ddot{Q})$
$\therefore \varrho(S(S\ddot{Q}), S\ddot{Q}, S\ddot{Q}) = 0$
$\therefore S(S\ddot{Q}) = \ddot{Q}$
Since S is one-one follows that $S\ddot{Q} = \ddot{Q}$
$\therefore T\ddot{Q} = S\ddot{Q} = \ddot{Q}$
Let μ, τ be mothers of $S$and T.
$\therefore S\mu = T\mu = \mu$and$S\tau = T\tau = \tau$
Suppose $\mu \neq \tau$
Then

$$\varrho(\mu, \mu, \tau) = \varrho(S\mu, S\mu, S\tau) > \varrho(T\mu, T\mu, T\tau) = \varrho(\mu, \mu, \tau)$$

$\Rightarrow \varrho(\mu, \mu, \tau) < \varrho(\mu, \mu, \tau)$which is a contradiction.
$\therefore \mu = \tau$
Thus S and T have only one common mother.
The following Example is in support of

Theorem-3.6.

Example 3.7. Let $\Theta = \left[0, \frac{1}{2}\right], \Omega = \left[0, \frac{1}{2}\right]$ and $\Psi = \left[0, \frac{1}{2}\right]$and $\Theta \cap \Omega \cap \Psi = (0,0,0)$.
Define $\varrho_1: \Theta \times \Omega \times \Psi \to [0, +\infty)$ be defined by $\varrho_1(\Theta, \Omega, \Psi) = |\hat{\theta}| + |\hat{\omega}| + |\hat{\psi}|$.Then $(\Theta, \Omega, \Psi, \varrho_1)$ be a complete stalk space by Example- (2.6). Let $S, T: (\Theta, \Omega, \Psi) \Rightarrow (\Theta, \Omega, \Psi)$ be two stalk maps defined by $S\theta = \theta^2, T\theta = \theta^3$. Then S and T satisfy all the hypothesis of Theorem-3.6.
Consider $S, T: (\Theta, \Omega, \Psi) \Rightarrow (\Theta, \Omega, \Psi)$ be two stalk maps defined by $S\theta = \theta^2$and $T\theta = \theta^3$.
If $\theta \in \Theta$ then $S(\theta, 0,0) = (\theta^2, 0,0)$ and $T(\theta, 0,0) = (\theta^3, 0,0)$
If $\theta \in \Omega$ then $S(0, \theta, 0) = (0, \theta^2, 0)$ and $T(0, \theta, 0) = (0, \theta^3, 0)$
If $\theta \in \Psi$ then $S(0,0, \theta) = (0,0, \theta^2)$ and $T(0,0, \theta) = (0,0, \theta^3)$
(1):$\Rightarrow S(\Theta, \Omega, \Psi) = T(\Theta, \Omega, \Psi)$
$\Rightarrow \theta^2 = \theta^3, \omega^2 = \omega^3and\psi^2 = \psi^3$
$\Rightarrow \theta = 0, \omega = 0and\psi = 0$
$\Rightarrow (ST)(\Theta, \Omega, \Psi) = S(\theta^3, \omega^3, \psi^3)$
$= (\theta^6, \omega^6, \psi^6) = (TS)(\Theta, \Omega, \Psi)$
$\therefore S$and$T$ are weakly compatible.
(2):$\Rightarrow S(\Theta) = \left[0, \frac{1}{4}\right] \times \{0\} \times \{0\}$,
$S(\Omega) = \{0\} \times \left[0, \frac{1}{4}\right] \times \{0\}$
and$S(\Psi) = \{0\} \times \{0\} \times \left[0, \frac{1}{4}\right]$
$\therefore S\Theta \subseteq \Theta, S\Omega \subseteq \Omega$and $S\Psi \subseteq \Psi$.
Let $(\theta_n, \omega_n, \psi_n) \in \Theta \times \Omega \times \Psi$ and $(S\theta_n, S\omega_n, S\psi_n) \to 0$
$\Rightarrow (\theta_n^2, \omega_n^2, \psi_n^2) \to 0$
$\Rightarrow \varrho_1(\theta_n^2, \omega_n^2, \psi_n^2) = \hat{\theta}_n^2 + {\hat{\omega}_n^2}^2 + \hat{\psi}_n^2 \to 0$
and(0,0,0) $\in$ core.
$S(\theta_n, \omega_n, \psi_n) \to (0,0,0)$
$\therefore S(\theta_n, \omega_n, \psi_n) \to S(0,0,0)$
$\therefore$ Condition (3.6.2) holds.
(3): Claim: S is one-one.
$S(\theta, \omega, \psi) = S(\theta', \omega', \psi')$
$\Rightarrow (\theta^2, \omega^2, \psi^2) = (\theta'^2, \omega'^2, \psi'^2)$
$\Rightarrow \theta^2 = \theta'^2, \omega^2 = \omega'^2and\psi^2 = \psi'^2$
$\Rightarrow \theta = \theta', \omega = \omega'and\psi = \psi'$
where $\theta, \omega, \psi, \theta', \omega', \psi' \in \left[0, \frac{1}{2}\right]$.
$\therefore$ Condition (3.6.3) holds.
(4):$\Rightarrow T\Theta = \left[0, \frac{1}{8}\right]andS\Theta = \left[0, \frac{1}{4}\right]$
$\therefore T\Theta \subseteq S\Theta$
In a simlar way we have,
$T\Omega \subseteq S\Omega$and$T\Psi \subseteq S\Psi$.
$\therefore$ Condition (3.6.4) holds.
(5): Claim: $\epsilon \leqslant \varrho(S\theta, S\omega, S\psi) < \epsilon + \delta \Rightarrow \varrho(T\theta, T\omega, T\psi) < \epsilon$.
Let $\delta = \epsilon^2$ and $\varrho(S\theta, S\omega, S\psi) = \hat{\theta}^2 + \hat{\omega}^2 + \hat{\psi}^2$
Suppose $\hat{\theta}^3 + \hat{\omega}^3 + \hat{\psi}^3 \geqslant \epsilon$.

$$\begin{aligned}
\epsilon < \hat{\theta}^2 + \hat{\omega}^2 + \hat{\psi}^2 &< \epsilon + \epsilon^2 \\
&< \hat{\theta}^3 + \hat{\omega}^3 + \hat{\psi}^3 + \left(\hat{\theta}^3 + \hat{\omega}^3 + \hat{\psi}^3\right)^2 \\
&\leqslant \left(\hat{\theta}^3 + \hat{\omega}^3 + \hat{\psi}^3\right) + \left(\hat{\theta}^3 + \hat{\omega}^3 + \hat{\psi}^3\right)\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) \\
&= \left(\hat{\theta}^3 + \hat{\omega}^3 + \hat{\psi}^3\right)\left(1 + \frac{3}{8}\right) \\
&= \frac{11}{8}\left(\hat{\theta}^3 + \hat{\omega}^3 + \hat{\psi}^3\right) \\
&< \frac{11}{8}\left(\hat{\theta}^2\left(\frac{1}{2}\right) + \hat{\omega}^2\left(\frac{1}{2}\right) + \hat{\psi}^2\left(\frac{1}{2}\right)\right) \\
&< \frac{11}{8}\left(\hat{\theta}^2 + \hat{\omega}^2 + \hat{\psi}^2\right) < \left(\hat{\theta}^2 + \hat{\omega}^2 + \hat{\psi}^2\right)
\end{aligned}$$

which is a contradiction
$\varrho(T\theta, T\omega, T\psi) = \hat{\theta}^3 + \hat{\omega}^3 + \hat{\psi}^3 < \epsilon$.
$\therefore$ Condition (3.6.5) holds.
Clearly (0,0,0) is the only mother of $S$and T.

IV. Conclusion and future work

We introduced stalk spaces, stalk maps and establish fixed point theorems in stalk spaces for stalk maps.We have given some examples which are supporting to our results. We study the notion of continuity of stalk maps.We explored a new contractive condition namely ST-condition for stalk maps. In future we will extend our results to some more conditions.

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