TOTAL EDGE-CORDIAL LABELING OF KOCH CURVE AND KOCH SNOWFLAKE GRAPHS

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Abstract

A graph G(V, E) is said to be edge-cordial if it is possible to label the edges with the set {0, 1}, with the induced vertex labeling $f(v)$ computed by $f(v) = \sum_{uv \in E} f(uv) \pmod 2$ for each $v \in V$, such that $|E_f(0) - E_f(1)| \le 1$ and $|V_f(0) - V_f(1)| \le 1$, where $E_f(i)$ and $V_f(i)$, i = 0, 1, are the number of edges and vertices labeled with 0 and 1 respectively. An edge-cordial labeling in which the number of vertices and edges labeled with 0 and the number of vertices and edges labeled with 1 differ by at most 1 (i.e) $\left|\left(V_f(0) + E_f(0)\right) - \left(V_f(1) + E_f(1)\right)\right| \le 1$ is called as a total edge-cordial labeling. In this paper, we have examined the existence of edge-cordial and total edge cordial labeling of the fractal graphs derived from Koch-Curve and Koch Snowflake.

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1. Introduction

By a *graph G* we mean a finite, undirected, connected graph without any loops or multiple edges. Let *V(G)* and *E(G)* be the set of vertices and edges of a graph *G*, respectively. The order and size of a graph G is denoted as $p = |V(G)|$ and $q = |E(G)|$ respectively. For general graph theoretic notions we refer Harrary [3].

By a labeling we mean a one-to-one mapping that carries a set of graph elements onto a set of numbers (usually positive or non-negative integers), called labels. There are several types of labeling and a detailed survey of many of them can be found in the dynamic survey of graph labeling by J.A. Gallian [4].
Cahit [1] has introduced a variation of both graceful and harmonious labeling and named it as cordial labeling, which is defined as follows:

Let f be a function from the vertices of G to {0, 1} and for each edge *uv* assign the label $|f(u) - f(v)|$. Then f is called a *cordial labeling* of G if (i) $\left|V_f(0) - V_f(1)\right| \le 1$ and (ii) $\left|E_f(0) - E_f(1)\right| \le 1$ where $\left|V_f(0)\right|$ is the number of vertices labeled 0, $\left|V_f(1)\right|$ is the number of vertices labeled 1, $\left|E_f(0)\right|$ is the number of edges labeled 0 and $\left|E_f(1)\right|$ is the number of edges labeled 1. If there exists a cordial labeling for a graph G then it is called a cordial graph [1, 2].

In [2] Cahit proved the following: every tree is cordial; K_n is cordial if and only if $n \le 3$; $K_{m,n}$ is cordial for all m and n; the friendship graph $C_3^{(t)}$ (i.e., the one-point union of t 3-cycles) is cordial if and only if $t \ne 2$ (mod 4); all fans are cordial; the wheel W_n is cordial if and only if $n \ne 3$ (mod 4); maximal outer planar graphs are cordial; and an Eulerian graph is not cordial if its size is congruent to 2 (mod 4).

Yilmaz and Cahit [12] introduced edge-cordial labeling as a binary edge labeling $f : E(G) \to \{0, 1\}$, with the induced vertex labeling given by $f(v) = \sum_{uv \in E} f(uv) \pmod 2$ for each $v \in V$, such that $\left|E_f(0) - E_f(1)\right| \le 1$ and $\left|V_f(0) - V_f(1)\right| \le 1$, where $E_f(i)$ and $V_f(i)$, i = 0, 1, denote the number of edges and vertices labeled with 0 and 1 respectively. They also proved the following lemmas.

Lemma 1.1. [12] If a labeling f of any graphs satisfies $\left|E_f(0) - E_f(1)\right| \le 1$, then $V_f(0) \equiv 0 \pmod 2$.

Lemma 1.2. [12] A graph is edge-cordial only when $p \ne 2 \pmod 4$, where p is the order of G.

In [10] Samir K. Vaidya and Chirag M. Barasara have introduced the notion of edge product cordial labeling. They defined an edge product cordial labeling of a graph G with vertex set V as a function f from V to {0, 1} such that if each edge uv is assigned the label f(u)f(v), the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1, and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1.
Also in [11] they have introduced the notion of total product cordial labeling. They defined a total product cordial labeling of a graph G with vertex set V as a function f from V to {0, 1} such that if each edge uv is assigned the label f(u)f(v) the number of vertices and edges labeled with 0 and the number of vertices and edges labeled with 1 differ by at most 1. A graph with a total product cordial labeling is called a total product cordial graph.

As an extension of the above, in a total edge-cordial labeling of a graph G with vertex set V and edge set E is defined as an edge-cordial labeling such that number of vertices and edges labeled with 0 and the number of vertices and edges labeled with 1 differ by at most 1 (i.e) $\left|\left(V_f(0) + E_f(0)\right) - \left(V_f(1) + E_f(1)\right)\right| \le 1$. A graph with a total edge-cordial labeling is called a *total edge-cordial graph*. The existence of this labeling for several classes of graphs was discussed in [6, 7, 8, 9].

A fractal [5] is an object or quantity that displays self-similarity in a somewhat technical sense, on all scales. The object need not exhibit exactly the same structure at all scales, but the same "type" of structures must appear on all scales.

A Koch curve which was named after Helge von Koch in 1904 is a fractal which is constructed as follows: Begin with a straight line of unit length and divide it into three equally sized parts. The middle section is replaced with an equilateral triangle and its base is removed. After first iteration, the length is increased by four-thirds and the process is repeated.

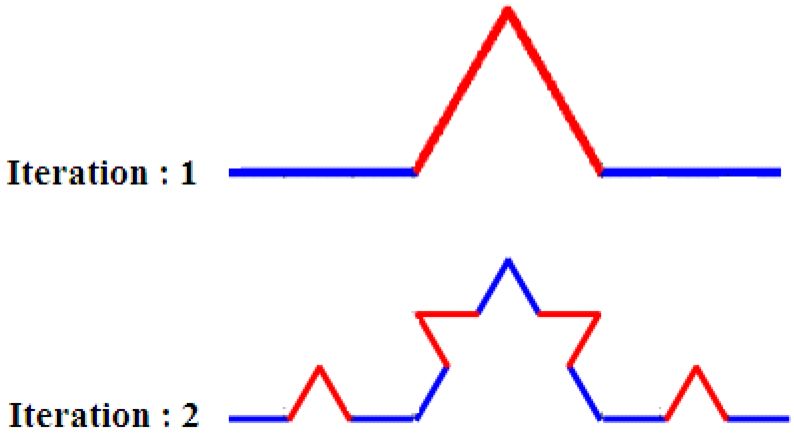

Fig. 1. Koch Curve

Similar to the above, the Koch Snowflake is generated in very much the same way as the Koch Curve. The only variation is that, rather than using a line of unit length as the initial figure, an equilateral triangle is used. This fractal, if magnified three times in any area and also displays the property of self-similarity.

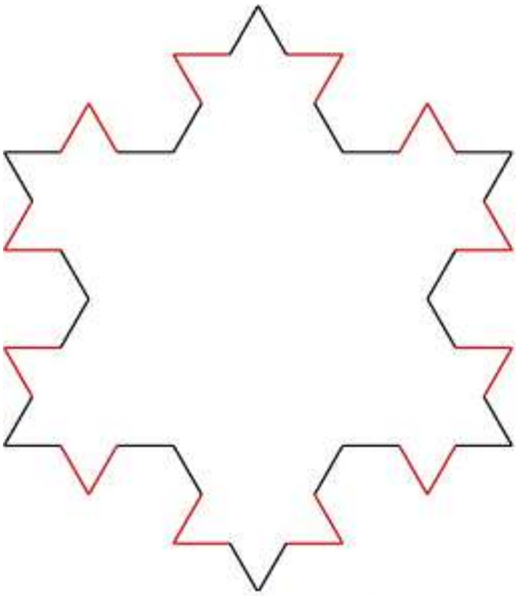

Fig.2. Koch Snowflake

In this paper, we examine the existence of edge-cordial and total edge cordial labeling of the fractal graphs derived from Koch-Curve and Koch Snowflake.

2. Main Results

Here we consider Koch curve as a graph KC_n , $n \geq 1$ where n is the number of iteration used. Since KC_n consist of four self similar copies of KC_{n-1} , $n \geq 2$ we get $|V| = 4^n + 1$ and $|E| = 4^n$ where V and E are the set of vertices and edges of KC_n , $n \geq 1$ respectively.

Also we consider Koch Snowflake as a graph KS_n , $n \geq 1$ where n is the number of iteration used. Since KS_n consist of three self similar copies of KC_n , $n \geq 1$ and each of the end point of first copy of common for the second, etc. we have $|V| = 3(4^n)$ and $|E| = 3(4^n)$ where V and E are the set of vertices and edges of KS_n , $n \geq 1$ respectively.

First we prove the existence of edge-cordial and total edge-cordial labeling of paths and cycles.

Lemma 2.1. *Every path graph* P_n , $n \neq 2(\ mod\ 4\ )$ *is edge-cordial.*

Proof: Let the vertex set and edge set of P_n , $n \geq 5$ be given by $V = \{v_1 , v_2 , ..., v_n\}$ and $E = \{v_i v_{i+1} / 1 \leq i \leq n-1\}$.

By lemma 1.1 it is clear that P_n is not edge-cordial when $n \equiv 2(\ mod\ 4\ )$

Let us define an edge labeling f : E → {0, 1}, as follows:

Case (i) : $n \equiv 0(\ mod\ 4\ )$ or $n \equiv 1(\ mod\ 4\ )$

For $1 \leq i \leq n-1$, define

$$f(v_i v_{i+1}) = \begin{cases} 0 & if\ i \equiv 1\ or\ 2\ (\text{mod } 4) \\ 1 & if\ i \equiv 0\ or\ 3\ (\text{mod } 4) \end{cases}$$

Then we get $$E(0) = \begin{cases} \frac{n}{2} & if\ n\ is\ even \\ \frac{n-1}{2} & if\ n\ is\ odd \end{cases}$$

and $$E(1) = \begin{cases} \frac{n}{2} - 1 & if\ n\ is\ even \\ \frac{n-1}{2} & if\ n\ is\ odd \end{cases}.$$

Thus we have

$$|E(0) - E(1)| = \begin{cases} 1 & if\ n\ is\ even \\ 0 & if\ n\ is\ odd \end{cases}.$$

Case (ii) : $n \equiv 3(\ mod\ 4\ )$

For $1 \leq i \leq n-1$, define

$$f(v_i v_{i+1}) = \begin{cases} 0 & if\ 1 \le i \le \frac{n+1}{4} \\ 1 & if\ \frac{n+3}{2} \le i \le n-1 \\ 1 & if\ i = \frac{n+1}{4}+1, \frac{n+1}{4}+3, \ldots, \frac{n+1}{4}+\frac{n-5}{2} \\ 0 & if\ i = \frac{n+1}{4}+2, \frac{n+1}{4}+4, \ldots, \frac{n+1}{4}+\frac{n-3}{2} \end{cases}$$

Then we get $E(0) = \frac{n-1}{2}$ and $E(1) = \frac{n-1}{2}$.

Thus we have $\left| E(0) - E(1) \right| = 0$.

In both the cases we observe that $\left| E(0) - E(1) \right| \le 1$.

Also if we define $g(v) = \sum_{uv \in E} f(uv) \,(mod\ 2)$ for each $v \in V$,

Then we get,

$$V(0) = \begin{cases} \frac{n}{2} & if\ n \equiv 0\ (\text{mod}\ 4) \\ \frac{n+1}{2} & if\ n \equiv 1\ (\text{mod}\ 4) \\ \frac{n-1}{2} & if\ n \equiv 3\ (\text{mod}\ 4) \end{cases}$$

and

$$V(1) = \begin{cases} \frac{n}{2} & if\ n \equiv 0\ (\text{mod}\ 4) \\ \frac{n-1}{2} & if\ n \equiv 1\ (\text{mod}\ 4) \\ \frac{n+1}{2} & if\ n \equiv 3\ (\text{mod}\ 4) \end{cases}$$

Thus we have $\left| V(0) - V(1) \right| \le 1$.

Hence P_n, $n \ne 2(mod\ 4)$ is edge-cordial. □

Lemma 2.2. *Every cycle graph* C_n, $n \ne 2(mod\ 4)$ *is edge-cordial.*

Proof: Let the vertex set and edge set of $C_n, n \ge 4$ be given by $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{v_i v_{i+1} / 1 \le i \le n-1\} \cup \{v_n v_1\}$.

By lemma 1.1 it is clear that C_n is not edge-cordial when $n \equiv 2(mod\ 4)$

Let us define an edge labeling $f : E \rightarrow \{0, 1\}$, as follows:

For $1 \le i \le n-1$, define

$$f(v_i v_{i+1}) = \begin{cases} 0 & \text{if } i \equiv 1 \text{ or } 2 \pmod 4 \\ 1 & \text{if } i \equiv 0 \text{ or } 3 \pmod 4 \end{cases}$$

and

$$f(v_n v_1) = \begin{cases} 0 & \text{if } n \equiv 1 \pmod 4 \\ 1 & \text{if } n \equiv 0 \text{ or } 3 \pmod 4 \end{cases}$$

Then we get, $E(0) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$

and $E(1) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$.

Thus we have

$$|E(0) - E(1)| = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}.$$

$\Rightarrow |E(0) - E(1)| \leq 1$.

Also if we define $g(v) = \sum_{uv \in E} f(uv) \pmod 2$ for each $v \in V$,

Then we get,

$$V(0) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 4 \\ \frac{n+1}{2} & \text{if } n \equiv 1 \pmod 4 \\ \frac{n-1}{2} & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

and

$$V(1) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 4 \\ \frac{n-1}{2} & \text{if } n \equiv 1 \pmod 4 \\ \frac{n+1}{2} & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

Thus we have $|V(0) - V(1)| \leq 1$.

Hence C_n , $n \neq 2(\ mod\ 4\ )$ is edge-cordial. □

As a direct implication of lemmas 2.1 and 2.2, we get the following results.

Lemma 2.3. *Every path graph* P_n , $n \neq 2(\ mod\ 4\ )$ *is total edge-cordial.* □

Lemma 2.4. *Every cycle graph* C_n , $n \neq 2(\ mod\ 4\ )$ *is total edge-cordial.* □

Observation 2.5. From the graph representation of the Koch Curve, it is clear that the graph KC_n , $n \geq 1$ is isomorphic to the path graph P_{4n^2+1} .

Observation 2.6. From the graph representation Koch Snowflake, it is clear that the graph KS_n , $n \geq 1$ is isomorphic to the cycle graph C_{4n^2} .

Theorem 3.3.1. *Every Koch curve graph* KC_n , $n \geq 1$ *is edge-cordial.*

Proof: The proof follows directly from lemma 2.1, as the Koch curve graph KC_n , $n \geq 1$ is isomorphic to P_{4n^2+1} and $4n^2+1 \neq 2\ (\text{mod } 4)$. □

Theorem 3.3.2. *Every Koch snowflake graph* KS_n , $n \geq 1$ *is edge-cordial.*

Proof: The proof follows directly from lemma 2.2, as the Koch snowflake graph KS_n , $n \geq 1$ is isomorphic to C_{4n^2} and $4n^2+1 \neq 2\ (\text{mod } 4)$. □

As a direct implication of lemmas 2.3 and 2.4, we get the following results.

Theorem 3.3.3. *Every Koch curve graph* KC_n , $n \geq 1$ *is total edge-cordial.* □

Theorem 3.3.4. *Every Koch snowflake graph* KS_n , $n \geq 1$ *is total edge-cordial.* □

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