THE LATTICE OF CONVEX SUBLATTICE OF $S(S(B_n))$

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Abstract

Subbarayan.R and Vethamanickam.A[15] have proved in their paper that $CS(B_n)$ the lattice of convex sublattices of a Boolean algebra B_n, of rank n, with respect to the set inclusion relation, is a dual simplicial Eulerian lattice. Subsequently, Sheeba Merlin.G and Vethamanickam.A[8] have proved in their paper that $CS[S(B_n)]$ is an Eulerian lattice under the set inclusion relation which is neither simplicial nor dual simplicial. In this paper, we prove that $CS[S(S(B_n))]$ is an Eulerian lattice under the set inclusion relation and it is neither simplicial nor dual simplicial, if$n > 1$.

Keywords: Convex sublattice; Simplicial Eulerian lattice; Dual simplicial.

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1 Introduction

The study of lattice of convex sublattices of a lattice was started by K. M. Koh[3], in the year 1972. He had investigated the internal structure of a lattice$L$, in relation to $CS(L)$, like so many other authors for various algebraic structures such as groups, Boolean algebras, directed graphs and so on. A construction of a new Eulerian lattice $S(B_n)$ from a Boolean algebra B_n of rank n is found in the thesis of V. K. Santhi[12] in 1992.

In 2012, R.Subbarayan and A.Vethamanickam[15] have proved in their paper that the lattice of convex sublattices of a Boolean algebra B_n, of rank n , $CS(B_n)$ with respect to the set inclusion relation is a dual simplicial Eulerian lattice. In 2017, Sheeba Merlin.G and Vethamanickam.A[8] proved in their paper that $CS(S(B\_n)$ is an Eulerian lattice under the set inclusion relation which is neither simplicial nor dual simplicial. In this paper, we are going to look at the structure of

$CS\left(S(S(B_n))\right)$ and prove it to be Eulerian under $' \subseteq '$ relation. $S(B_3)$ is shown in figure 1. We note that $S(B_3)$ contains three copies of B_3, we call them left copy, right copy and middle copy of $S(B_3)$.

2 Preliminaries

Throughout this section $CS(L)$, the collection of all convex sublattices of a lattice L including empty set is equipped with the partial order of set inclusion relation.

Definition 2.1 Möbius function

The **Möbius** function μ on a finite graded poset P is an integer-valued function defined on $P \times P$

by the formulae: $\mu(x,y) = \begin{Bmatrix} 1, if x = y; \\ 0, if x \nless y; \\ \sum_{-x \le z < y} \mu(x,z), if x < y \end{Bmatrix}$

An equivalent definition for an Eulerian poset is as follows:

Definition 2.2 Eulerian poset

A finite graded poset P is said to be *Eulerian* if its Möbius function assumes the value

$$\mu(x,y) = (-1)^{r(y)-r(x)} \forall x \le y \ in \ P$$

Lemma 2.3 [8]

A finite graded poset P is Eulerian if and only if all intervals $[x,y]$ of length $\boldsymbol{P}$ contain an equal number of elements of odd and even rank.

Definition 2.5 Simplicial

A poset P is called *Simplicial* if for all $t \neq 1$ in P, $[0,t]$ is a Boolean algebra and P

is called *Dual Simplicial* if for all $t \neq 0$ in P ,$[t,1]$ is a Boolean algebra.

Lemma 2.6[1]

Let L and K be any two lattices. Then $CS(L \times K) \cong [(CS(L) - \phi \times (CS(K) - \phi)] \cup \phi$.

Lemma 2.7 [15]

Let B_n be a Boolean lattice of rank n . Then $CS(B_n)$ is a dual simplicial Eulerian lattice.

We note that any interval of an Eulerian lattice is Eulerian and an Eulerian lattice cannot contain a three element chain as an interval. For any undefined term we refer to[2], [11]and [12].

3 The Eulerian property of the lattice $\boldsymbol{CS}(\boldsymbol{S}(\boldsymbol{S}(B_n))$

Lemma

For $n \geq 1$, we have

$1+\left[\binom{n}{1}+2\right]+\left[2\binom{n}{1}+\binom{n}{2}\right]+\left[2\binom{n}{2}+\binom{n}{3}\right]+\cdots+2\binom{n}{n-2}+\binom{n}{n-1}+2\binom{n}{n-1}+1=$ $3.2^n-2.$

Theorem

$CS[S(S(B_n))]$, the lattice of convex sublattices of $S(S(B_n))$ with respect to the set inclusion relation is an Eulerian lattice.

Proof.

We first note that, the number of elements of ranks $0,1,2,\cdots,n+1$ in $S(B_n)$ are, $1,2+\binom{n}{1},2\binom{n}{1}+\binom{n}{2},2\binom{n}{2}+\binom{n}{3}\},\cdots,2\binom{n}{n-2}+\binom{n}{n-1}\},2\binom{n}{n-1}$, 1 respectively.

The number of elements of ranks $0,1,2,\cdots,n+2$ in $S[S(B_n)]$ are, $1,2+\binom{n}{1},2+\binom{n}{1}+2,2\binom{n}{1}+2]+2\binom{n}{1}+\binom{n}{2},2\left[2\binom{n}{1}+\binom{n}{2}\right]+2\binom{n}{2}+\binom{n}{3},2\left[2\binom{n}{2}+\binom{n}{3}\right]+2\binom{n}{3}+\binom{n}{4},\cdots,2[2\binom{n}{n-2}+\binom{n}{n-12}]+2\binom{n}{n-1},2[2\binom{n}{n-1}]$,1 respectively.

It is clear that the rank of $CS[S(S(B_n))]$, is $n+3$.

We are going to prove that $CS[S(S(B_n))]$ is Eulerian.

That is, to prove that this interval $[\phi,S(S(B_n))]$ has the same number of elements of odd and even rank.

Let A_i be the number of elements of rank i in $CS[S(S(B_n))]$, $i=1,2,\cdots,n+2$.

A_1 = The number of singleton convex sublattices of $S[S(B_n)]$

$=1+2+\binom{n}{1},+2+\binom{n}{1}+2+2\binom{n}{1}+2]+2\binom{n}{1}+\binom{n}{2}+2\left[2\binom{n}{1}+\binom{n}{2}\right]+2\binom{n}{2}+\binom{n}{3}+2\left[2\binom{n}{2}+\binom{n}{3}\right]+2\binom{n}{3}+\binom{n}{4}+\cdots+2\left[2\binom{n}{n-2}+\binom{n}{n-12}\right]+2\binom{n}{n-1}+2\left[2\binom{n}{n-1}\right]+1$ …….(3.2.1)

A_2 = The number of elements of rank 2 in $S[S(B_n)]$

=The number of edges in $S[S(B_n)]$

=The number of edges containing 0+ number of edges with an atom at the bottom + the number of edges from the rank 2 elements$+\cdots+$ the number of edges with a coatom of $S[S(B_n)]$ at the bottom.

Number of edges containing 0 is $2+\binom{n}{1}+2$

Number of edges with an extreme atom at the bottom= $\binom{n}{1}+2$ There are 2 extreme atoms, therefore total number of such edges $=2\left[\binom{n}{1}+2\right]$.

From the atom of the left copy of middle copy, the number of edges= $2\left[\binom{n}{1}+2\right]$. There are totally $2\left[\binom{n}{1}+2\right]$edges from the extreme atoms of the middle copy.

Now, to find the number of edges from an atom of the middle of the middle copy.

Let x be an atom in the middle copy, then $[x,1]\simeq S\left[S\left(B_{\{n-1\}}\right)\right]$

Therefore, the total number of edges from an atom at the middle copy $= 2 + \binom{n-1}{1} + 2$. There are totally $\binom{n}{1}$ atoms in the middle of the middle copy.

Therefore, the number of edges with an atom at the bottom in the middle of the middle copy$\binom{n}{1}\left[2 + \binom{n-1}{1} + 2\right]$.

Hence, the number of edges with an atom at the bottom is $2\left[\binom{n}{1} + 2\right] + 2\left[\binom{n}{1} + 2\right] + \binom{n}{1}\left[2 + \binom{n-1}{1} + 2\right]$.

Now to find, the number of edges with an element of rank 2 at the bottom.

Let x be a rank 2 element in the left copy. Then$[x, 1] \simeq$
$$\begin{cases} B_n \ if \ x \in \ extreme \ copies \ of \ left \ copy \ of \ S(S(B_n)) \\ S\left(B_{\{n-1\}}\right) if \ x \in \ middle \ copy \ of \ left \ copy \ of \ S(S(B_n)) \end{cases}$$

If $[x, 1] \simeq B_n$, There are $\binom{n}{1}$ edges in both extreme copies. Totally, $2\binom{n}{1}$ edges are there.

If $[x, 1] \simeq S(B_{\{n-1\}})$, the number of edges from x is $\binom{n-1}{1} + 2$.There are$\binom{n}{1}$such elements, therefore, totally $\binom{n}{1}[\binom{n-1}{1} + 2]$ edges in the middle of the left copy of $S\left(S(B_n)\right)$.Therefore, the number of edges with an element of rank 2 at the bottom in the left copy $= 2\binom{n}{1} + \binom{n}{1}[\binom{n-1}{1} + 2]$.Similarly, the number of edges with an element of rank 2 at the bottom in the right copy $= 2\binom{n}{1} + \binom{n}{1}\left[\binom{n-1}{1} + 2\right]$.

Let x be a rank 2 element in the middle copy.

Then, $[x, 1] \simeq \begin{cases} S\left(B_{\{n-1\}}\right) \ if \ x \in \ extreme \ copies \ of \ middle \ copy \ of \ S(S(B_n)) \\ S(S\left(B_{\{n-2\}}\right) if \ x \in \ middle \ copy \ of \ middle \ copy \ of \ S(S(B_n)) \end{cases}$

If $[x, 1] \simeq S\left(B_{\{n-1\}}\right)$, the number of edges from x is $\binom{n-1}{1} + 2$. There are $2\binom{n}{1}$ such elements in both extreme copies. Totally, $2\binom{n}{1}$ $[\binom{n-1}{1} + 2$] edges.

If $[x, 1] \simeq S(S(B_{\{n-2\}}))$,the number of edges from x is $2 + \binom{n-2}{1} + 2$. There are $\binom{n}{2}$ such elements, therefore, totally $\binom{n}{2}$ $[2 + \binom{n-2}{1} + 2]$ edges in the middle of the middle copy of $S(S(B_n))$. Therefore, the number of edges with an element of rank 2 at the bottom in the middle copy is $2[\binom{n}{1}(\binom{n-1}{1} + 1)] + \binom{n}{2}[2 + \binom{n-2}{1} + 2]$ edges.

Hence, total number of edges from a rank 2 element is $2\left[2\binom{n}{1} + \binom{n}{1}\left[\binom{n-1}{1} + 2\right]\right] + 2\binom{n}{1}\left[\binom{n-1}{1} + 2\right] + \binom{n}{2}\left[2 + \binom{n-2}{1} + 2\right]$.

Now to find, the number of edges with an element of rank 3 at the bottom.

Let x be a rank 3 element in the extreme copies in the left copy of $S\left(S(B_n)\right)$.

$$[x, 1] \simeq \begin{cases} B_{n-1} \ if \ x \in \ extreme \ copies \ of \ left \ copy \ of \ S(S(B_n)) \\ S\left(B_{\{n-2\}}\right) if \ x \in \ middle \ copy \ of \ left \ copy \ of \ S(S(B_n)) \end{cases}$$

If $[x,1] \simeq B_{\{n-1\}}$, the number of edges from x is $\binom{n-1}{1}$.There are $2\binom{n}{1}$ such x's in both extreme copies.Totally, $2\binom{n}{1}(\binom{n-1}{1})$ edges from such x's in the extreme copies of left copy.

If $[x,1] \simeq S(B_{\{n-2\}})$, then the number of edges from x is $\binom{n-2}{1}+2$

There are $\binom{n}{2}$ such elements x, therefore, totally $\binom{n}{2}[\binom{n-2}{1}+2]$ edges from x's in middle of the left copy of $S(S(B_n))$. Therefore, the number of edges with an element of rank 3 at the bottom in the left copy is, $2\binom{n}{1}\binom{n-1}{1}+\binom{n}{2}\left[\binom{n-2}{1}+2\right]$. Similarly, the total number of edges from a rank 3 element in the right copy is $2\binom{n}{1}\binom{n-1}{1}+\binom{n}{2}\left[\binom{n-2}{1}+2\right]$.

Let x be a rank 3 element in the middle copy of $(S(B_n))$.

$$[x,1] \simeq \begin{cases} S(B_{\{n-2\}}) \; if \; x \in \; extreme \; copies \; of \; middle \; copy \; of \; S(S(B_n)) \\ S(S(B_{\{n-3\}})if \; x \in \; middle \; copy \; of \; middle \; copy \; of \; S(S(B_n)) \end{cases}$$

If $[x,1] \simeq S(B_{\{n-2\}})$, the number of edges from x is $\binom{n-2}{1}+2$. There are $2\binom{n}{2}$ such elements in both extreme copies. Totally, $2\binom{n}{2}$ $[\binom{n-1}{1}+2$] edges.

If $[x,1] \simeq S(S(B_{\{n-2\}}))$,the number of edges from x is $2+\binom{n-3}{1}+2$. There are $\binom{n}{3}$ such elements, therefore, totally $\binom{n}{3}$ $[2+\binom{n-3}{1}+2]$ edges in the middle of the middle copy of $S(S(B_n))$. Therefore, the number of edges with an element of rank 3 at the bottom in the middle copy is $2[\binom{n}{2}(\binom{n-2}{1}+1)]+\binom{n}{3}[2+\binom{n-3}{1}+2]$ edges.

Hence, total number of edges from a rank 3 element is $2\{2\binom{n}{1}\binom{n-1}{1}+\binom{n}{2}\left[\binom{n-2}{1}+2\right]\}+2[\binom{n}{2}(\binom{n-2}{1}+1)]+\binom{n}{3}[2+\binom{n-3}{1}+2]$.

Let x be a rank 4 element in the extreme copies in the left copy of $S(S(B_n))$.

$$[x,1] \simeq \begin{cases} B_{n-2} \; if \; x \in \; extreme \; copies \; of \; left \; copy \; of \; S(S(B_n)) \\ S(B_{\{n-3\}})if \; x \in \; middle \; copy \; of \; left \; copy \; of \; S(S(B_n)) \end{cases}$$

If $[x,1] \simeq B_{\{n-2\}}$, the number of edges from x is $\binom{n-2}{1}$.There are $2\binom{n}{2}$ such x's in both extreme copies.Totally, $2\binom{n}{2}(\binom{n-2}{1})$ edges from such x's in the extreme copies of left copy.

If $[x,1] \simeq S(B_{\{n-3\}})$, then the number of edges from x is $\binom{n-3}{1}+2$

There are $\binom{n}{3}$ such elements x, therefore, totally $\binom{n}{3}[\binom{n-3}{1}+2]$ edges from x's in middle of the left copy of $S(S(B_n))$. Therefore, the number of edges with an element of rank 3 at the bottom in the left copy is, $2\binom{n}{2}(\binom{n-2}{1})+\binom{n}{3}[\binom{n-3}{1}+2]$. Similarly, the total number of edges from a rank 3 element in the right copy is $2\binom{n}{2}(\binom{n-2}{1})+\binom{n}{3}[\binom{n-3}{1}+2]$.

Let x be a rank 4 element in the middle copy of $(S(B_n))$.

$$[x,1] \simeq \begin{cases} S(B_{\{n-3\}}) \; if \; x \in \; extreme \; copies \; of \; middle \; copy \; of \; S(S(B_n)) \\ S(S(B_{\{n-4\}})if \; x \in \; middle \; copy \; of \; middle \; copy \; of \; S(S(B_n)) \end{cases}$$

If $[x,1] \simeq S(B_{\{n-3\}})$, the number of edges from x is $\binom{n-3}{1}+2$. There are $2\binom{n}{3}$ such elements in both extreme copies. Totally, $2\binom{n}{3}\left[\binom{n-3}{1}+2\right]$ edges.

If $[x,1] \simeq S(S(B_{\{n-4\}}))$,the number of edges from x is $2+\binom{n-4}{1}+2$. There are $\binom{n}{4}$ such elements, therefore, totally $\binom{n}{4}\left[2+\binom{n-4}{1}+2\right]$ edges in the middle of the middle copy of $S(S(B_n))$. Therefore, the number of edges with an element of rank 4 at the bottom in the middle copy is $2\binom{n}{3}\left[\binom{n-3}{1}+2\right]+\binom{n}{4}\left[2+\binom{n-4}{1}+2\right]$ edges.

Hence, total number of edges from a rank 4 element is$2\{2\binom{n}{2}(\binom{n-2}{1})+\binom{n}{3}[\binom{n-3}{1}+2]\}+2\binom{n}{3}\left[\binom{n-3}{1}+2\right]+\binom{n}{4}\left[2+\binom{n-4}{1}+2\right]$.

Hence, we get, the total number of edges in $S(S(B_n))$ is,$A_2=2+\binom{n}{1}+2+2\left[\binom{n}{1}+2\right]+2\left[\binom{n}{1}+2\right]+\binom{n}{1}\left[2+\binom{n-1}{1}+2\right]+2\left[2\binom{n}{1}+\binom{n}{1}\left[\binom{n-1}{1}+2\right]\right]+2\binom{n}{1}\left[\binom{n-1}{1}+2\right]+\binom{n}{2}\left[2+\binom{n-2}{1}+2\right]+2\left\{2\binom{n}{1}\binom{n-1}{1}+\binom{n}{2}\left[\binom{n-2}{1}+2\right]\right\}+2\left[\binom{n}{2}\left(\binom{n-2}{1}+1\right)\right]+\binom{n}{3}\left[2+\binom{n-3}{1}+2\right]+2\left\{2\binom{n}{2}\left(\binom{n-2}{1}\right)+\binom{n}{3}\left[\binom{n-3}{1}+2\right]\right\}+2\binom{n}{3}\left[\binom{n-3}{1}+2\right]+\binom{n}{4}\left[2+\binom{n-4}{1}+2\right]+\cdots+2\left\{2\left[2\binom{n}{n-2}+\binom{n}{n-1}\right]+2\binom{n}{n-1}\right\}+4\binom{n}{n-1}$ ……..(3.2.2)

A_3 = The number of 4 element convex sublattices in $S[S(B_n)]$

= The number of B_2's in $S[S(B_n)]$

=The number of 4 element convex sublattices containing 0+ number of 4 element convex sublattices containing an atom at the bottom + the number of 4 element convex sublattices containing a rank 2 element at the bottom + ⋯ + The number of 4 element convex sublattices containing a rank n element at the bottom in $S(S(B_n))$.

The number of 4 element convex sublattices in $S(S(B_n))$ containing 0 as the bottom element is, $2\left(\binom{n}{1}+2\right)+2\binom{n}{1}+\binom{n}{2}$.

Next, we find the number of 4 element convex sublattices containing an atom as the bottom element.

Fix an atom ∈ $S(S(B_n))$, If x is the bottom element of the left copy of $S(S(B_n))$, then $[x,1] \simeq S(B_n)$.

Therefore, the number of B_2's containing x at the bottom is $2\binom{n}{1}+\binom{n}{2}$. Similarly, The number of B_2's containing the bottom element of the right copy is$2\binom{n}{1}+\binom{n}{2}$.

Let x be an atom in the middle copy, then $[x,1] \simeq \begin{cases} S(B_n)\ if\ x \in\ extreme\ copies\ of\ middle\ copy\ of\ S(S(B_n)) \\ S(S(B_{\{n-1\}}))if\ x \in\ middle\ copy\ of\ middle\ copy\ of\ S(S(B_n)) \end{cases}$

If $[x,1] \simeq S(B_{\{n\}})$, the number of B_2's from an atom at the middle copy $=2\binom{n}{1}+\binom{n}{2}$. There are 2 extreme copies. Totally,$2(2\binom{n}{1}+\binom{n}{2})$. $B_2's$.

If $[x,1] \simeq S\left(S\left(B_{\{n-1\}}\right)\right)$, the number of B_2's from x is $2\left[\binom{n-2}{1}+2\right]+2\binom{n-2}{1}+\binom{n-2}{2}$.

There are $\binom{n}{1}$ such elements, therefore, totally $\binom{n}{1}$ $[2\left[\binom{n-2}{1}+2\right]+2\binom{n-2}{1}+\binom{n-2}{2}]$ edges in the middle of the middle copy of $S(S(B_n))$.

Therefore, the total number of B_2's from an atom at the bottom $= 2\left\{2\binom{n}{1}+\binom{n}{2}\right\}+\binom{n}{1}$ $[2\left[\binom{n-2}{1}+2\right]+2\binom{n-2}{1}+\binom{n-2}{2}]$. Totally, $2\left\{2\binom{n}{1}+\binom{n}{2}\right\}+2\left\{2\binom{n}{1}+\binom{n}{2}\right\}+\binom{n}{1}$ $[2\left[\binom{n-2}{1}+2\right]+2\binom{n-2}{1}+\binom{n-2}{2}]$ $B_2's$.

Now to find, the number of B_2's s with an element of rank 2 at the bottom.

Let x be a rank 2 element in the left copy. Then$[x,1] \simeq$
$$\begin{cases} B_n\ if\ x \in\ extreme\ copies\ of\ left\ copy\ of\ S(S(B_n)) \\ S\left(B_{\{n-1\}}\right) if\ x \in\ middle\ copy\ of\ left\ copy\ of\ S(S(B_n)) \end{cases}$$

If $[x,1] \simeq B_n$, There are $\binom{n}{2}$ B_2's in both extreme copies. Totally, $2\binom{n}{2}$ B_2's are there.

If $[x,1] \simeq S\left(B_{\{n-1\}}\right)$, the number of B_2's from x is $2\binom{n-1}{1}+\binom{n-1}{2}$.There are$\binom{n}{1}$such elements, therefore, totally $\binom{n}{1}\left[2\binom{n-1}{1}+\binom{n-1}{2}\right]$ B_2's in the middle of the left copy of $S\left(S(B_n)\right)$.

Therefore, the number of B_2's with an element of rank 2 at the bottom in the left copy $= 2\binom{n}{2}+\binom{n}{1}[2\binom{n-1}{1}+\binom{n-1}{2}]$.Similarly, the number of B_2's with an element of rank 2 at the bottom in the right copy $= 2\binom{n}{2}+\binom{n}{1}[2\binom{n-1}{1}+\binom{n-1}{2}]$.

Let x be a rank 2 element in the middle copy.

$$\text{Then, } [x,1] \simeq \begin{cases} S\left(B_{\{n-1\}}\right)\ if\ x \in\ extreme\ copies\ of\ middle\ copy\ of\ S(S(B_n)) \\ S(S\left(B_{\{n-2\}}\right) if\ x \in\ middle\ copy\ of\ middle\ copy\ of\ S(S(B_n)) \end{cases}$$

If $[x,1] \simeq S\left(B_{\{n-1\}}\right)$, the number of B_2's from x is $2\binom{n-1}{1}+\binom{n-1}{2}$. There are 2 such extreme copies. Totally, $2\{\ \binom{n}{1}[2\binom{n-1}{1}+\binom{n-1}{2}]\ \}$edges.

If $[x,1] \simeq S\left(S\left(B_{\{n-2\}}\right)\right)$,the number of B_2's from x is $2\binom{n-2}{1}+2+2\binom{n-2}{1}+\binom{n-1}{1}$. There are $\binom{n}{2}$ such elements, therefore, totally $\binom{n}{2}$ $[2\binom{n-2}{1}+2+2\binom{n-2}{1}+\binom{n-1}{1}]$ edges in the middle of the middle copy of $S(S(B_n))$. Therefore, the number of B_2's with an element of rank 2 at the bottom in the middle copy is $2\{\ \binom{n}{1}[2\binom{n-1}{1}+\binom{n-1}{2}]\ \}+\binom{n}{2}[2\binom{n-2}{1}+2+2\binom{n-2}{1}+\binom{n-1}{1}]$ edges.

Hence, total number of B_2's from a rank 2 element is $2\left[2\binom{n}{2}+\binom{n}{1}[2\binom{n-1}{1}+\binom{n-1}{2}]\right]+$ $2\{\ \binom{n}{1}[2\binom{n-1}{1}+\binom{n-1}{2}]\ \}+\binom{n}{2}[2\binom{n-2}{1}+2+2\binom{n-2}{1}+\binom{n-1}{1}]$.

Now to find, the number of B_2's with an element of rank 3 at the bottom.

Let x be a rank 3 element in the extreme copies in the left copy of $S\left(S(B_n)\right)$.

$$[x,1] \simeq \begin{cases} B_{n-1} \; if \; x \in \; extreme \; copies \; of \; left \; copy \; of \; S(S(B_n)) \\ S(B_{\{n-2\}}) if \; x \in \; middle \; copy \; of \; left \; copy \; of \; S(S(B_n)) \end{cases}$$

If $[x,1] \simeq B_{\{n-1\}}$, the number of B_2's from x is $\binom{n-1}{2}$.There are $2\binom{n-1}{2}$ such x's in both extreme copies.Totally, $2\binom{n-1}{2}$ B_2's from such x's in the extreme copies of left copy.

If $[x,1] \simeq S(B_{\{n-2\}})$, then the number of B_2's from x is $2\binom{n-2}{1} + \binom{n-2}{2}$

There are $\binom{n}{2}$ such elements x, therefore, totally $\binom{n}{2}[2\binom{n-2}{1} + \binom{n-2}{2}]$ B_2's from x's in middle of the left copy of $S(S(B_n))$. Therefore, the number of B_2's with an element of rank 3 at the bottom in the left copy is, $2\binom{n}{1}\binom{n-1}{2} + \binom{n}{2}\left[2\binom{n-2}{1} + \binom{n-2}{2}\right]$. Similarly, the total number of B_2's from a rank 3 element in the right copy is $2\binom{n}{1}\binom{n-1}{2} + \binom{n}{2}\left[2\binom{n-2}{1} + \binom{n-2}{2}\right]$.

Let x be a rank 3 element in the middle copy of $(S(B_n))$.

$$[x,1] \simeq \begin{cases} S(B_{\{n-2\}}) \; if \; x \in \; extreme \; copies \; of \; middle \; copy \; of \; S(S(B_n)) \\ S(S(B_{\{n-3\}}) if \; x \in \; middle \; copy \; of \; middle \; copy \; of \; S(S(B_n)) \end{cases}$$

If $[x,1] \simeq S(B_{\{n-2\}})$, the number of B_2's from x is $2\binom{n-2}{1} + \binom{n-2}{2}$. There are $2\binom{n}{2}$ such elements in both extreme copies. Totally, $2\{\binom{n}{2}\ [2\binom{n-2}{1} + \binom{n-2}{2}\ ]\}$ edges.

If $[x,1] \simeq S(S(B_{\{n-2\}}))$,the number of B_2's s from x is $2\left[\binom{n-3}{1} + 2\right] + 2\binom{n-3}{1} + \binom{n-3}{2}$. There are $\binom{n}{3}$ such elements, therefore, totally $\binom{n}{3}\ [2\left[\binom{n-3}{1} + 2\right] + 2\binom{n-3}{1} + \binom{n-3}{2}]$ edges in the middle of the middle copy of $S(S(B_n))$. Therefore, the number of B_2's with an element of rank 3 at the bottom in the middle copy is $2\{\binom{n}{2}\ [2\binom{n-2}{1} + \binom{n-2}{2}\ ]\} + \binom{n}{3}\ [2\left[\binom{n-3}{1} + 2\right] + 2\binom{n-3}{1} + \binom{n-3}{2}]$ edges.

Hence, total number of B_2's from a rank 3 element is $2\left\{2\binom{n}{1}\binom{n-1}{2} + \binom{n}{2}\left[2\binom{n-2}{1} + \binom{n-2}{2}\right]\right\} + 2\left\{\binom{n}{2}\left[2\binom{n-2}{1} + \binom{n-2}{2}\right]\right\} + \binom{n}{3}\left[2\left[\binom{n-3}{1} + 2\right] + 2\binom{n-3}{1} + \binom{n-3}{2}\right]$.

Hence, $A_3 = 2\left(\binom{n}{1} + 2\right) + 2\binom{n}{1} + \binom{n}{2} + 2\left\{2\binom{n}{1} + \binom{n}{2}\right\} + \binom{n}{1}\left[2\left[\binom{n-2}{1} + 2\right] + 2\binom{n-2}{1} + \binom{n-2}{2}\right] + 2\left\{2\binom{n}{1} + \binom{n}{2}\right\} + 2\left\{2\binom{n}{1} + \binom{n}{2}\right\} + \binom{n}{1}\left[2\left[\binom{n-2}{1} + 2\right] + 2\binom{n-2}{1} + \binom{n-2}{2}\right] + 2\left[2\binom{n}{2} + \binom{n}{1}\left[2\binom{n-1}{1} + \binom{n-1}{2}\right]\right] + 2\left\{\binom{n}{1}\left[2\binom{n-1}{1} + \binom{n-1}{2}\right]\right\} + \binom{n}{2}\left[2\binom{n-2}{1} + 2 + 2\binom{n-2}{1} + \binom{n-1}{1}\right] + 2\left\{2\binom{n}{1}\binom{n-1}{2} + \binom{n}{2}\left[2\binom{n-2}{1} + \binom{n-2}{2}\right]\right\} + 2\left\{\binom{n}{2}\left[2\binom{n-2}{1} + \binom{n-2}{2}\right]\right\} + \binom{n}{3}\left[2\left[\binom{n-3}{1} + 2\right] + 2\binom{n-3}{1} + \binom{n-3}{2}\right] + \cdots + 2\left[2\binom{n}{n-2} + \binom{n}{n-1}\right] + 2\binom{n}{n-1}$.

Similar argument will give, $A_4 = 2\left(2\binom{n}{1} + \binom{n}{2}\right) + 2\binom{n}{2} + \binom{n}{3} + 2\left\{2\binom{n}{2} + \binom{n}{3}\right\} + \binom{n}{1}\left[2\left[2\binom{n-1}{1} + \binom{n-1}{2}\right] + 2\binom{n-1}{2} + \binom{n-1}{3}\right] + 2\left\{2\binom{n}{3} + \binom{n}{2}\right\}\left[2\left[\binom{n-2}{1} + 2\right] + 2\binom{n-2}{1} + \binom{n-2}{2}\right] + 2\left[2\binom{n}{2} + \binom{n}{1}\left[2\binom{n-1}{1} + \binom{n-1}{2}\right]\right] + 2\left\{\binom{n}{1}\left[2\binom{n-1}{1} + \binom{n-1}{2}\right]\right\} + \binom{n}{2}\left[2\binom{n-2}{1} + 2 + 2\binom{n-2}{1} + \binom{n-1}{1}\right] + 2\left\{2\binom{n}{1}\binom{n-1}{2} + \binom{n}{2}\left[2\binom{n-2}{1} + \binom{n-2}{2}\right]\right\} + 2\left\{\binom{n}{2}\left[2\binom{n-2}{1} + \binom{n-2}{2}\right]\right\} + \binom{n}{3}\left[2\left[\binom{n-3}{1} + 2\right] + 2\binom{n-3}{1} + \binom{n-3}{2}\right] + \cdots + 2\left[2\binom{n}{n-2} + \binom{n}{n-1}\right] + \binom{n}{n-1}$ and so on.

Finally, we get $A_{n+2} = 2\left(2\binom{n}{n-1}\right) + 2 + \binom{n}{1} + 2$

Case (i): Suppose that n is even.

Therefore, $n + 3$ is odd. $A_1 - A_2 + A_3 - \cdots - A_n + A_{\{n+1\}} - A_{\{n+2\}} + 0$

Case(ii):

Suppose that n is odd,

Therefore, $n + 3$ is even.

$$A_1 - A_2 + A_3 - \cdots + A_n - A_{\{n+1\}} + A_{\{n+2\}} = 2.$$

Though in the above theorem we have proved that $CS[S(S(B_n))]$ is Eulerian, it is neither Simplicial nor dual simplicial.

$CS[S(S(B_n))]$ is not dual simplicial since, the upper interval $[1, S(S(B_n))]$ in $CS[S(S(B_n))]$ contains $4\binom{n}{n-1}$ number of atoms which is greater than $n + 2$, the rank of $[1, S(S(B_n))]$, implying that $[1, S(S(B_n))]$, is not Boolean.

$CS\big[S\big(S(B_n)\big)\big]$ is not simplicial since, the lower interval $\big[\phi, [l_1, 1]\big]$ where l_1 is the left extreme atom of $S(S(B_n))$ contains $3.2^n - 2$ number of atoms by Lemma 3.1, which cannot be equal to $n + 2$, the rank of$\big[\phi, [l_1, 1]\big]$, implying that $\big[\phi, [l_1, 1]\big]$ is not Boolean.

Conclusions

In this paper, we have proved that $CS[S(S(B_n))]$ is an Eulerian lattice under the set inclusion

relation which is neither simplicial nor dual simplicial, $if\ n > 1$. We strongly believe that the result proved in this paper, can be extended to more general Eulerian lattices and any other general lattices.

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