Energy and spectrum of $G_{m,n}{}^{M}$graph

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Abstract: The notation of an undirected simple graph $G_{m,n}{}^{M} = (V, E)$ on a finite subset of natural numbers $m, n \in N$,where the vertex set $V = \{1,2, \dots . n\}$ and any two distinct vertices $u, v \in V$ are adjacent if and only if $u \neq v$and $u.v$ is not divisible by m. The energy of the graph is the summation of the absolute values of all eigen values of the adjacency matrix of a graph G.Matrix energy is the summation of all absolute singular values of graph G. In this paper, the computation of energy, matrix energy of the graph $G_{m,n}{}^{M}$ are discussed and the results are obtained.

Key words: Spectrum of a graph**,** Energy of a graph, Matrix energy of a graph.

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1. Introduction

The energy of the graph is introduced by Gutman[1] in 1978, π-electron energy is determined to identify the inside the Hückel atomic orbital approximation [2, 3] by the calculation of graph energy. The adjacency matrix of a graph G is denoted by $A(G)$ and is defined as $A(G) = \begin{Bmatrix} 1, & if\ v_i, v_j\ \ are\ adjacent\ in\ G \\ 0, & otherwise \end{Bmatrix}$. The eigen values of $A(G)\ of\ G$ are denoted by $\omega_1, \omega_{2,\dots}\omega_n$ where $\omega_1 \geq \omega_2 \geq \cdots \geq \omega_n$ .The spectral radius of ω_1 of G is the highest eigen value of G. The spectrum of the graph G is the collection of eigen values with their multiplicities of an adjacency matrix $A(G)$ is $\begin{pmatrix} \omega_1 & \dots . & \omega_n \\ m_1 & \dots . & m_n \end{pmatrix}$. The energy of a graph G is the sum of absolute eigen values of A(G) of $G.$ i.e.$E(G) = \sum_{i=1}^{n} |\omega_i|$. The applications of graph spectra was presented by D. Cvetkovi´c et al[4]. The matrix energy of G by observing the relationship between eigen values and singular values of an adjacency matrix of a G is extended by Nikiforov [5]. The undirected graph $G_{m,n}{}^{M}$ is introduced by Ivy Chakrabarthy et al[6] and proved some basic properties of $G_{m,n}{}^{M}$.

Motivated by the above work the authors studied the concepts of energy, matrix energy of $G_{m,n}{}^M$ graph at various values of n.The notations and terminology used in this paper found in [7].

2. $G_{m,n}{}^M$ Graph and its properties

Definition: The Undirected simple graph $G_{m,n}{}^M = (V, E)$ on a finite subset of natural numbers $m, n \in N$,where the vertex set $V = \{1,2, \dots . n\}$ and two distinct vertices $u, v \in V$ are adjacent if and only if $u \neq v$and $u.v$ is not divisible by m.

Lemma 2.1: Let $m = 1$ then the graph $G_{m,n}{}^M$ is a null graph with n vertices.

Lemma 2.2: For $1 < m \leq n$, the graph $G_{m,n}{}^M$ is disconnected.

Lemma 2.3: For $m > n$, the graph $G_{m,n}{}^M$is connected.

Lemma 2.4: The Maximum degree of the graph $G_{m,n}{}^M$ is $n - 1$.

3. Energy and Matrix Energy of $G_{m,n}{}^M$ graph

Let $G_{m,n}{}^M$be a simple graph with n vertices.Let $A(G_{m,n}{}^M)$ be the adjacency matrix of the graph $G_{m,n}{}^M$ is defined as $A(G_{m,n}{}^M) = \begin{Bmatrix} 1, & if\ v_i, v_j\ \ are\ adjacent\ in\ G_{m,n}{}^M \\ 0, & otherwise \end{Bmatrix}$and $\omega_1, \omega_{2,\dots}\omega_n$ are the eigenvalues of $A(G_{m,n}{}^M)$ where $\omega_1 \geq \omega_2 \geq \cdots \geq \omega_n$. The spectra of the graph $G_{m,n}{}^M$ is the eigen values with their corresponding multiplicities of $A(G_{m,n}{}^M)$of the graph $G_{m,n}{}^M$ is $\begin{pmatrix} \omega_1 & \dots . & \omega_n \\ m_1 & \dots . & m_n \end{pmatrix}$. The energy of the graph $G_{m,n}{}^M$is the sum of absolute eigen values of an adjacency matrix $A(G_{m,n}{}^M)$ of a graph $G_{m,n}{}^M$.That is $E\left(G_{m,n}{}^M\right) = \sum_{i=1}^{n} |\omega_i|$.

Let $A(G_{m,n}{}^M)A(G_{m,n}{}^M)'$is a positive semi definite matrix where $A(G_{m,n}{}^M)'$is the transpose of $A(G_{m,n}{}^M)$. Let $\mu_1, \mu_{2,\dots}\mu_n$are the singular values of $A(G_{m,n}{}^M)$and these are the square root values of eigen values of $A\left(G_{m,n}{}^M\right)A(G_{m,n}{}^M)'$where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$.Now the summation of absolute singular values of $A(G_{m,n}{}^M)$is defined as the matrix energy of the graph$G_{m,n}{}^M$. That is $E_m\left(G_{m,n}{}^M\right) = \sum_{i=1}^{n} |\mu_i|$.

Theorem 3.1: The energy of the graph $G_{m,n}{}^M$ when $n = 2p, p$ is prime,$m > n, m$ is prime is $2(2p - 1)$.

Proof: By the definition of the graph $G_{m,n}{}^M$, the vertex set V is defined as $\{1,2, \dots . n\}$ when $n = 2p, m > n,\ m, p$ are prime. Then the adjacency matrix of the graph $G_{m,n}{}^M$ is

$$A\left(G_{m,2p}{}^{M}\right) = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}_{2p\times 2p}$$

The Characteristic Equation of $A(G_{m,2p}{}^{M})$ of the graph $G_{m,2p}{}^{M}$ is $(\omega+1)^{2p-1}(\omega-(2p-1)) = 0$.

Then -1 *and* $(2p-1)$ are the eigen values of $A(G_{m,2p}{}^{M})$ and their corresponding multiplicities are $(2p-1)$ and 1. Hence the spectrum of the graph $G_{m,2p}{}^{M}$ is $\begin{pmatrix} -1 & 2p-1 \\ 2p-1 & 1 \end{pmatrix}$.

The energy of the graph $G_{m,2p}{}^{M}$ is $E(G_{m,2p}{}^{M}) = |-1|(2p-1) + |2p-1|(1) = 2(2p-1)$.

Theorem 3.2: The matrix energy of the graph $G_{m,n}{}^{M}$ when $n = 2p, p$ is prime, $m > n, m$ is prime is $2(2p-1)$.

Proof: From the Theorem 3.1, the adjacency matrix of the graph $G_{m,2p}{}^{M}$ is

$$A\left(G_{m,2p}{}^{M}\right) = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}_{2p\times 2p}$$

Then $A\left(G_{m,2p}{}^{M}\right)A\left(G_{m,2p}{}^{M}\right)' = \begin{pmatrix} T & U \\ U & T \end{pmatrix}_{2p\times 2p}$

Where $T = \begin{pmatrix} 2p-1 & 2p-2 & 2p-2 & \dots & 2p-2 \\ 2p-2 & 2p-1 & 2p-2 & \dots & 2p-2 \\ 2p-2 & 2p-2 & 2p-1 & \dots & 2p-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2p-2 & 2p-2 & 2p-2 & \dots & 2p-1 \end{pmatrix}_{p\times p}$ and

$$U = (2p-2)\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}_{p\times p}$$

The Characteristic Equation of $A(G_{m,2p}{}^{M})A\left(G_{m,2p}{}^{M}\right)'$ of the graph $G_{m,2p}{}^{M}$ is

$(\omega-1)^{2p-1}(\omega-(2p-1)^2)=0.$

Then $1,(2p-1)$ are the singular values of $A(G_{m,2p}{}^{M})$and their corresponding multiplicities are $(2p-1)$ and 1. Hence the spectrum of the graph $G_{m,2p}{}^{M}$ is $\begin{pmatrix} 1 & 2p-1 \\ 2p-1 & 1 \end{pmatrix}$.

The matrix energy of the graph $G_{m,2p}{}^{M}$ is $E(G_{m,2p}{}^{M}) = |1|(2p-1)+|2p-1|(1) = 2(2p-1)$.

Theorem 3.3: The energy of the graph $G_{m,n}{}^{M}$ when $m>n, m$ is prime, n is prime is $2(n-1)$.

Proof: By the definition of the graph $G_{m,n}{}^{M}$, the vertex set V is defined as when m and n are primes and $m>n$ is $V=\{1,2,\dots.n\}$.

Then the adjacency matrix of the graph $G_{m,n}{}^{M}$ is

$$A\left(G_{m,n}{}^{M}\right) = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}_{n\times n}$$

The Characteristic Equation of $A(G_{m,n}{}^{M})$ of the graph $G_{m,n}{}^{M}$ is $(\omega+1)^{n-1}(\omega-(n-1))=0$.

Then $-1,(n-1)$ are the eigen values of $A(G_{m,n}{}^{M})$and their corresponding multiplicities are $(n-1)$ and 1. Hence the spectrum of the graph $G_{m,n}{}^{M}$ is $\begin{pmatrix} -1 & n-1 \\ n-1 & 1 \end{pmatrix}$.

The energy of the graph $G_{m,n}{}^{M}$ is $E(G_{m,n}{}^{M}) = |-1|(n-1)+|n-1|(1) = 2(n-1)$.

Theorem 3.4: The matrix energy of the graph $G_{m,n}{}^{M}$ when $m>n, m$ is prime, n is prime is $2(n-1)$.

Proof: By the definition of the graph $G_{m,n}{}^M$, the vertex set V is defined as $\{1,2,\ldots.n\}$, when m and n are primes and $m > n$.

From Theorem 3.3 the adjacency matrix of the graph $G_{m,n}{}^M$ is

$$A(G_{m,n}{}^M) = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}_{n\times n}$$

Then $A(G_{m,n}{}^M)A(G_{m,n}{}^M)' = \begin{pmatrix} T & U \\ U & T \end{pmatrix}_{n\times n}$

where $T = \begin{pmatrix} n-1 & n-2 & n-2 & \dots & n-2 \\ n-2 & n-1 & n-2 & \dots & n-2 \\ n-2 & n-2 & n-1 & \dots & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n-2 & n-2 & n-2 & \dots & n-1 \end{pmatrix}_{\frac{n}{2}\times\frac{n}{2}}$ and $U = (n-2)\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}_{\frac{n}{2}\times\frac{n}{2}}$

The Characteristic Equation of $A(G_{m,n}{}^M)A(G_{m,n}{}^M)'$ of the graph $G_{m,n}{}^M$ is

$(\omega - 1)^{n-1}(\omega - (n-1)^2) = 0.$

Then $1, (n-1)$ are the singular values of $A(G_{m,n}{}^M)$ and their corresponding multiplicities are $(n-1)$ and 1. Hence the spectrum of the graph $G_{m,n}{}^M$ is $\begin{pmatrix} 1 & n-1 \\ n-1 & 1 \end{pmatrix}$.

The matrix energy of the graph $G_{m,n}{}^M$ is $E_m(G_{m,n}{}^M) = |1|(n-1) + |n-1|(1) = 2(n-1)$.

Theorem 3.5: The energy of the graph $G_{m,n}{}^M$ where $m = n, m > 1$ is prime and n prime is $2(n-1)$.

Proof: By the definition of the graph $G_{m,n}{}^M$, the vertex set V is defined as $\{1,2,\ldots.n\}$, when m and n are primes and $m > 1$.

Then the adjacency matrix of the graph $G_{m,n}{}^M$ is $A(G_{m,n}{}^M) = \begin{pmatrix} R & S & 0 \\ S & R & 0 \\ 0 & 0 & 0 \end{pmatrix}_{n\times n}$

$$\text{Where } R = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}_{\frac{n-1}{2}\times\frac{n-1}{2}} \text{ and } S = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}_{\frac{n-1}{2}\times\frac{n-1}{2}}$$

The Characteristic Equation of $A(G_{m,n}{}^{M})$ of $G_{m,n}{}^{M}$ is $\omega(\omega+1)^{n-1}(\omega-(n-1))=0$.

Then $0, -1, (n-1)$ are the eigen values of $A(G_{m,n}{}^{M})$and their corresponding multiplicities are $1, (n-1)$ and 1. Hence the spectrum of the graph $G_{m,n}{}^{M}$ is $\begin{pmatrix} 0 & -1 & n-1 \\ 1 & n-1 & 1 \end{pmatrix}$.

The energy of the graph $G_{m,n}{}^{M}$ is $E(G_{m,n}{}^{M}) = |0|(1) + |-1|(n-1) + |n-1|(1) = 2(n-1)$.

Theorem 3.6: The matrix energy of the graph $G_{m,n}{}^{M}$ where $m = n, m > 1$ is prime and n prime is $2(n-1)$.

Proof: By the definition of the graph $G_{m,n}{}^{M}$, the vertex set V is defined as $\{1,2,\dots.n\}$, when m and n are primes and $m > 1$.

From Theorem 3.5, the adjacency matrix of the graph $G_{m,n}{}^{M}$ is $A(G_{m,n}{}^{M}) = \begin{pmatrix} R & S & 0 \\ S & R & 0 \\ 0 & 0 & 0 \end{pmatrix}_{n\times n}$

$$\text{Where } R = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}_{\frac{n-1}{2}\times\frac{n-1}{2}} \text{ and } S = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}_{\frac{n-1}{2}\times\frac{n-1}{2}}$$

Then $A\left(G_{m,n}{}^{M}\right)A\left(G_{m,n}{}^{M}\right)' = \begin{pmatrix} T & U & 0 \\ U & T & 0 \\ 0 & 0 & 0 \end{pmatrix}_{n\times n}$

$$\text{Where } T = \begin{pmatrix} n-2 & n-3 & n-3 & \dots & n-3 \\ n-3 & n-2 & n-3 & \dots & n-3 \\ n-3 & n-3 & n-2 & \dots & n-3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n-3 & n-3 & n-3 & \dots & n-2 \end{pmatrix}_{\frac{n-1}{2}\times\frac{n-1}{2}} \text{ and } U = (n-3)\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}_{\frac{n-1}{2}\times\frac{n-1}{2}}$$

The Characteristic Equation of $A\left({G_{m,n}}^M\right)A\left({G_{m,n}}^M\right)'$ of the graph ${G_{m,n}}^M$ is $\omega(\omega-1)^{n-2}(\omega-(n-2)^2)=0$.

Then $0,1,(n-2)$ are the singular values of $A({G_{m,n}}^M)$and their corresponding multiplicities are $1,(n-2)$ and 1. Hence the spectrum of the graph ${G_{m,n}}^M$ is $\begin{pmatrix} 0 & 1 & n-2 \\ 1 & n-2 & 1 \end{pmatrix}$.

The matrix energy of the graph ${G_{m,n}}^M$ is $E_m({G_{m,n}}^M)$ = $|0|(1)+|1|(n-2)+|n-2|(1)=2(n-2)$.

Theorem 3.7: The energy of the graph ${G_{m,n}}^M$ where is $n=2^{\alpha},\alpha>1,m>n$ and m prime is $2(n-1)$.

Proof: Proof: By the definition of the graph ${G_{m,n}}^M$,the vertex set V is defined as when m is prime and $m>n$,$n=2^{\alpha}$ is $V=\{1,2,\dots.n\}$.

Then the adjacency matrix of the graph ${G_{m,2^{\alpha}}}^M$ is $A\left({G_{m,2^{\alpha}}}^M\right)=\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}_{2^{\alpha}\times 2^{\alpha}}$

The Characteristic Equation of $A({G_{m,2^{\alpha}}}^M)$ of ${G_{m,2^{\alpha}}}^M$ is $(\omega+1)^{2^{\alpha}-1}(\omega-(2^{\alpha}-1)=0$.

Then $-1,(2^{\alpha}-1)$ are the eigen values of $A({G_{m,2^{\alpha}}}^M)$and their corresponding multiplicities are $(2^{\alpha}-1)$ and 1. Hence the spectrum of the graph ${G_{m,2^{\alpha}}}^M$ is $\begin{pmatrix} -1 & 2^{\alpha}-1 \\ 2^{\alpha}-1 & 1 \end{pmatrix}$.

The energy of the graph ${G_{m,2^{\alpha}}}^M$ is $E({G_{m,2^{\alpha}}}^M)$ = $|-1|(2^{\alpha}-1)+|2^{\alpha}-1|(1)=2(2^{\alpha}-1)$.

Theorem 3.8: The matrix energy of the graph ${G_{m,n}}^M$ where is $n=2^{\alpha},\alpha>1,m>n$ and m prime is $2(n-1)$.

Proof: By the definition of the graph ${G_{m,n}}^M$, the vertex set V is defined as $\{1,2,\dots.n\}$, when m is prime and $m>n$,$n=2^{\alpha}$.

From Theorem 3.7, the adjacency matrix of the graph ${G_{m,2^{\alpha}}}^M$ is

$$A\left({G_{m,2^{\alpha}}}^{M}\right) = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}_{2^{\alpha}\times 2^{\alpha}}$$

Then $A\left({G_{m,2^{\alpha}}}^{M}\right)A\left({G_{m,2^{\alpha}}}^{M}\right)' = \begin{pmatrix} T & U \\ U & T \end{pmatrix}_{2^{\alpha}\times 2^{\alpha}}$

Where $T = \begin{pmatrix} 2^{\alpha}-1 & 2^{\alpha}-2 & 2^{\alpha}-2 & \dots & 2^{\alpha}-2 \\ 2^{\alpha}-2 & 2^{\alpha}-1 & 2^{\alpha}-2 & \dots & 2^{\alpha}-2 \\ 2^{\alpha}-2 & 2^{\alpha}-2 & 2^{\alpha}-1 & \dots & 2^{\alpha}-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2^{\alpha}-2 & 2^{\alpha}-2 & 2^{\alpha}-2 & \dots & 2^{\alpha}-1 \end{pmatrix}_{2^{\alpha-1}\times 2^{\alpha-1}}$

$$U = (2^{\alpha}-2)\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}_{2^{\alpha-1}\times 2^{\alpha-1}}$$

The Characteristic Equation of $A({G_{m,2^{\alpha}}}^{M})$ of ${G_{m,2^{\alpha}}}^{M}$ is $(\omega - 1)^{2^{\alpha}-1}(\omega - (2^{\alpha}-1)^{2}) = 0$.

Then $1, (2^{\alpha}-1)$ are the singular values of $A({G_{m,2^{\alpha}}}^{M})$and their corresponding multiplicities are $(2^{\alpha}-1)$ and 1. Hence the spectrum of the graph ${G_{m,2^{\alpha}}}^{M}$ is $\begin{pmatrix} 1 & 2^{\alpha}-1 \\ 2^{\alpha}-1 & 1 \end{pmatrix}$.

The energy of the graph ${G_{m,2^{\alpha}}}^{M}$ is $E_{m}({G_{m,2^{\alpha}}}^{M}) = |1|(2^{\alpha}-1) + |2^{\alpha}-1|(1) = 2(2^{\alpha}-1)$.

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