COEFFICIENT BOUNDS FOR DETERMINANT OF TOEPLITZ MATRICES CONTAINING BAZILEVIČ FUNCTIONS' COEFFICIENTS

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Abstract:
Our motive is to study the Toeplitz matrices which contains components that are the coefficients of Bazilevič functions. In the present paper, it has been tried to find the maximum values (upper bounds) for initial four Toeplitz determinants. Our results will generalize the recent papers of Radhika et al. [1] and Thomas et al. [2].

Keywords: Univalent functions, Analytic functions, q-derivative, Bazilevič functions, Toeplitz matrices.

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1. INTRODUCTION:

Suppose A is the class which contains functions having the format:

$$\zeta(z) = z + \sum_{i=2}^{\infty} a_i z^i, \tag{1.1}$$

and are also analytic in the open unit disk $\Delta = \{z: |z| < 1.\}$. Let S be the subclass of A containing univalent functions. Also, if $\zeta \in S$ then $\zeta \neq 0$ inside Δ . In this work, for $\zeta \in S$, the class $N(\chi)$ of Bazilevič functions of type χ; $0 \le \chi \le 1$ is considered, such that,

$$\Re\left(\frac{z^{1-\chi}\left(D_q\zeta\right)(z)}{\left[\zeta(z)\right]^{1-\chi}}\right) > 0,$$

where D_q is the q-derivative, which is given by:

$$\left(D_q\zeta\right)(z) = 1 + \sum_{i=2}^{\infty} [i]_q\, a_i z^{i-1}, \tag{1.2}$$

where

$$[i]_q = \frac{1-q^i}{1-q}, (q \neq 1).$$

Many researches (see [3], [4], [5], [6]) have studied different subfamilies of Bazilevič functions of type □ and investigated many properties including their initial coefficient bounds. Toeplitz matrices are well studied in recent years. These matrices arise in almost all the branches of statistics and probability, image processing, pure as well as applied mathematics, queueing networks, quantum mechanics, time series analysis and signal processing, (see [7]). Toeplitz matrices have computational properties and are used in large range of determinant computations and disparate algorithms.

The Toeplitz determinant (symmetric) is given by:

$$T_p(i) = \begin{vmatrix} a_i & a_{i+1} & \cdots & a_{i+p-1} \\ a_{i+1} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{i+p-1} & \cdots & \cdots & a_i \end{vmatrix}.$$

Here we gain the upper bounds for the above determinant $T_p(i)$; $p = 2,3; i = 1,2,3$. These elements of $T_p(i)$ are the coefficients of functions ζ (given by equation (1.1))belonging to the class of Bazilevič functions $N(\chi)$. To attain our main results, there will be use of the Lemma which is as follows:

Lemma 1.1 : [8] Let the function $\eta(z) = 1 + \sum_{n=1}^{\infty} \eta_n z^n \in P$. Hence, for any complex valued y and ε along with the conditions; $|y| \le 1$ as well as $|\varepsilon| \le 1$, we have,

$$2\eta_2 = \eta_1^2 + y(4-\eta_1^2),$$

$$4\eta_3 = \eta_1^3 + 2\left(4-\eta_1^2\right)\eta_1 y - \eta_1\left(4-\eta_1^2\right)y^2 + 2\left(4-\eta_1^2\right)\left(1-|y|^2\right)\varepsilon.$$

2. COEFFICIENT BOUNDS FOR TOEPLITZ DETERNINANTS:

In the following theorems, we find upper bounds for the coefficient bodies $T_2(2)$, $T_2(3)$, $T_3(1)$ and $T_3(2)$.

Theorem 2.1: Let ζ be the function which is given by equation (1.1) belongs tothe class $N(\chi)$. Then,

$$|T_2(2)| = |a_3^2 - a_2^2| \leq max.\left\{\frac{4}{\left(\chi+q+q^2\right)^2},\right.$$
$$\left.\frac{4[-\chi^4-\chi^3(4q+2q^2)-\chi^2(q^4+6q^3+6q^2-1)-\chi(2q^5+6q^4+4q^3-2q^2-4q)-q^6-2q^5+4q^3+4q^2]}{(\chi+q)^4(\chi+q+q^2)^2}\right\}. \tag{2.1}$$

Proof: If ζ is in the class N (χ) , then we have,

$$\frac{z^{1-\chi}\left(D_q\zeta\right)(z)}{\left[\zeta(z)\right]^{1-\chi}} = \eta(z).$$

By correlating the coefficients in equation given above, we get:

$$a_2 = \frac{\eta_1}{(\chi+q)}, \tag{2.2}$$

$$a_3 = \frac{\eta_2}{\left(\chi+q+q^2\right)} - \frac{(\chi-1)(\chi+2q)}{2(\chi+q)^2(\chi+q+q^2)}\eta_1^2. \tag{2.3}$$

Using equations (2.2) and (2.3), we obtain:

$$a_3^2 - a_2^2 = \frac{\eta_2^2}{\left(\chi+q+q^2\right)^2} + \frac{(\chi-1)^2(\chi+2q)^2\eta_1^4}{4(\chi+q)^4\left(\chi+q+q^2\right)^2} - \frac{(\chi-1)(\chi+2q)\eta_1^2\eta_2}{(\chi+q)^2\left(\chi+q+q^2\right)^2} - \frac{\eta_1^2}{(\chi+q)^2}. \tag{2.4}$$

By using lemma 1.1, it may be written that $2\eta_2 = \eta^2 + - y\left(4-\eta^2\right)$, hence with a constraining assumption we let $0\eta_1 \leq = \eta \leq 2$. Using this inequation (2.4), the quadratic equation in term of y is obtained.

$$a_3^2 - a_2^2 = \frac{\left(4-\eta^2\right)^2}{4\left(\chi+q+q^2\right)^2}y^2 + \frac{\left(\chi+2q+q^2\right)\eta^2\left(4-\eta^2\right)}{2(\chi+q)^2\left(\chi+q+q^2\right)^2}y +$$
$$\left(\frac{\left[\chi^2+\chi\left(2q^2+4q\right)+4q^2+4q^3+q^4\right]\eta^2 - 4(\chi+q)^2\left(\chi+q+q^2\right)^2}{4(\chi+q)^4\left(\chi+q+q^2\right)^2}\right)\eta^2. \tag{2.5}$$

Now applying the triangle inequality in above equation, we get:

$$|a_3^2 - a_2^2| \leq \frac{\left(4-\eta^2\right)^2}{4\left(\chi+q+q^2\right)^2} + \frac{\left(\chi+2q+q^2\right)\left(4-\eta^2\right)}{2(\chi+q)^2\left(\chi+q+q^2\right)^2}\eta^2 +$$
$$\left(\frac{\left[\chi^2+\chi\left(2q^2+4q\right)+4q^2+4q^3+q^4\right]\eta^2 - 4(\chi+q)^2\left(\chi+q+q^2\right)^2}{4(\chi+q)^4\left(\chi+q+q^2\right)^2}\right)\eta^2$$

$$= \phi(\eta, \chi).$$

Now differentiating ϕ (η , χ) with respect to η , we get:

$$\frac{\partial(\phi(\eta,\chi))}{\partial\eta}=\frac{-(4-\eta^2)\eta}{(\chi+q+q^2)^2}+\frac{2\eta(2-\eta^2)(\chi+2q+q^2)}{(\chi+q)^2(\chi+q+q^2)^2}+$$

$$\left(\frac{\left[\chi^2+\chi(2q^2+4q)+4q^2+4q^3+q^4\right]\eta^2+2(\chi+q)^2(\chi+q+q^2)^2}{(\chi+q)^4(\chi+q+q^2)^2}\right)\eta.$$

Or,

$$\frac{\partial(\phi(\eta,\chi))}{\partial\eta}=\eta\left\{\frac{\begin{array}{c}\eta^2\left[\chi^4+\chi^3(4q-2)+\chi^2(4q^2-8q+1)+\chi(4q-8q^2)+4q^2\right]-2\chi^4+\chi^3\\(4q^2-8q+4)+\chi^2(2q^4+12q^3-8q^2+16q)+\chi(4q^5+12q^4+20q^2)\\+8q^3+2q^4+4q^5+2q^6\end{array}}{(\chi+q)^4(\chi+q+q^2)^2}\right\}$$

Setting $\frac{\partial(\phi(\eta,\chi))}{\partial\eta}=0$ gives either $\eta=0$ or

$$\eta^2=\left\{\frac{2\chi^4-\chi^3(4q^2-8q+4)-\chi^2(2q^4+12q^3-8q^2+16q)-\chi(4q^5+12q^4+20q^2)-8q^3-2q^4-4q^5-2q^6}{\chi^4+\chi^3(4q-2)+\chi^2(4q^2-8q+1)+\chi(4q-8q^2)+4q^2}\right\}$$

But,

$$\left(2\chi^4-\chi^3(4q^2-8q+4)-\chi^2(2q^4+12q^3-8q^2+16q)-\chi(4q^5+12q^4+20q^2)-8q^3-2q^4-4q^5-2q^6\right)<0$$

for $0\le\chi\le1$. Consequently, the highest value of $\left|a_3^2-a_2^2\right|$ is gained at the terminal points, $\eta_1=\eta\in[0,2]$.

For $\eta_1=0$, we have $\eta_2=2y$. Therefore, from equation (2.4),

$$\left|a_3^2-a_2^2\right|=\frac{4|y|^2}{(\chi+q+q^2)^2}\le\frac{4}{(\chi+q+q^2)^2}.$$

For $\eta_1=2$, we have $a_2=\frac{2}{(\chi+q)}$ and $a_3=\frac{2}{(\chi+q+q^2)}-\frac{2(\chi-1)(\chi+2q)}{(\chi+q)^2(\chi+q+q^2)}$ which gives,

$$\left|a_3^2-a_2^2\right|\le$$

$$\left|\frac{4\left[-\chi^4-\chi^3(4q+2q^2)-\chi^2(q^4+6q^3+6q^2-1)-\chi(2q^5+6q^4+4q^3-2q^2-4q)-q^6-2q^5+4q^3+4q^2\right]}{(\chi+q)^4(\chi+q+q^2)^2}\right|$$

This in equality is sharp for the functions,

$$\frac{z^{1-\chi}(D_q\zeta)(z)}{[\zeta(z)]^{1-\chi}}=\frac{1+z}{1-z}.$$

Hence the theorem is proved.

Remark 2.2: For $q=1$, theorem 2.1 yields the bound for the class $B(\chi)$ of Bazilevič functions of type $\chi\,(0\le\chi\le1)$ given by Radhika et al. [1]. For $q=1$, $\chi=0$, theorem 2.1 gives the result obtained by Thomas et al. [2] which is about S^* (class of starlike functions). For $q=1$, $\chi=1$, the above theorem coincides with the result given by Sivasubramanian et al. [9] which is about. the class of functions having bounded boundary rotation.

Theorem 2.3: Let ζ be the function given by equation (1.1) belongs to the class $N(\chi), 0\le\chi\le1$. Then,

$$|T_2(3)|=|a_4^2-a_3^2|\le\max\left\{\frac{|R(\chi)|}{9(\chi+q)^6(\chi+q+q^2)^2(\chi+q+q^2+q^3)^2},\frac{4}{(\chi+q+q^2)^2}\right\},$$

$$\begin{aligned}R(\chi)=&4\chi^8+\chi^7(8q^2+32q)+\chi^6(4q^4+56q^3+112q^2-20)+\chi^5\left(24q^5+168q^4+168q^3-112q^2-160q\right)\\&+\chi^4\left(24q^6+96q^5-296q^4-664q^3-488q^2+16\right)\\&+\chi^3\left(-72q^8-336q^7-840q^6-1728q^5-1560q^4-720q^3+32q^2+128q\right)\\&+\chi^2\left(-36q^{10}-360q^9-1188q^8-2232q^7-2868q^6-1752q^5-356q^4+320q^3+448q^2\right)\\&+\chi(-72q^{11}-684q^{10}-1368q^9-2088q^8-1968q^7-792q^6+816q^5+960q^4+768q^3)\\&+(-36q^{12}-216q^{11}-504q^{10}-576q^9-144q^8+720q^7+1296q^6+1152q^5+576q^4).\end{aligned}$$

Proof: On correlating the coefficients of equation,

$$\frac{Z^{1-\chi}\left(D_q\zeta\right)(z)}{\left[\zeta(z)\right]^{1-\chi}}=\eta(z),$$

we get:

$$a_2=\frac{\eta_1}{\chi+q}, \tag{2.6}$$

$$a_3=\frac{\eta_2}{(\chi+q+q^2)}-\frac{(\chi-1)(\chi+2q)\eta_1^2}{2(\chi+q)^2(\chi+q+q^2)}, \tag{2.7}$$

and

$$a_4=\frac{\eta_3}{(\chi+q+q^2+q^3)}-\frac{(\chi-1)(\chi+2q+q^2)\eta_1\eta_2}{(\chi+q)(\chi+q+q^2)(\chi+q+q^2+q^3)}+$$
$$\frac{\left[2\chi^4+\chi^3\left(2q^2+8q-3\right)+\chi^2\left(3q^3+6q^2-12q+1\right)+\chi\left(4q-14q^2-3q^3\right)+6q^2\right]}{6(\chi+q)^3(\chi+q+q^2)(\chi+q+q^2+q^3)}\eta_1^3. \tag{2.8}$$

By using (2.7) and (2.8) and then by the application of Lemma 1.1, denoting $E=4-\eta^2$ and

$F=(1-|y|^2)\varepsilon$, where $|\varepsilon|<1$, we obtain:

$$a_4^2 - a_3^2$$

$$= \left[\frac{\begin{array}{l} \chi^8 + \chi^7(2q^2+8q) + \chi^6(q^4+14q^3+28q^2+4) + \chi^5(6q^5+42q^4+60q^3+8q^2+32q) + \\ \chi^4(15q^6+78q^5+106q^4+68q^3+112q^2+4) + \chi^3(24q^7+114q^6+180q^5+204q^4+216q^3 \\ +8q^2+32q) + \chi^2(27q^8+126q^7+228q^6+300q^5+280q^4+80q^3+112q^2) + \chi(18q^9+90q^8 \\ +192q^7+252q^6+384q^5+240q^4+192q^3) + 9q^{10}+54q^9+153q^8+288q^7+360q^6+288q^5+144q^4 \end{array}}{144(\chi+q)^6(\chi+q+q^2)^2(\chi+q+q^2+q^3)^2} \right] \eta_1^6$$

$$-\frac{\left[\chi^2+\chi\left(4q+2q^2\right)+4q^2+4q^3+q^4\right]}{4\left(\chi+q\right)^4\left(\chi+q+q^2\right)^2}\eta_1^4 + \frac{E^2F^2}{4(\chi+q+q^2+q^3)^2}$$

$$-\frac{\eta_1 y^2 E^2 F}{4\left(\chi+q+q^2+q^3\right)^2} + \frac{\left(\chi+2q+2q^2+q^3\right) yE^2\eta_1 F}{2\left(\chi+q\right)\left(\chi+q+q^2\right)\left(\chi+q+q^2+q^3\right)^2}$$

$$+\frac{[\chi^4+\chi^3\left(4q+q^2\right)+\chi^2\left(3q^3+6q^2+2\right)+\chi\left(8q+2q^2+6q^3+3q^4\right)+12q^2+12q^3+9q^4+3q^5]\eta_1^3 EF}{12\left(\chi+q\right)^3\left(\chi+q+q^2\right)\left(\chi+q+q^2+q^3\right)^2}$$

$$+\frac{\eta_1^2 E^2 y^4}{16\left(\chi+q+q^2+q^3\right)^2} - \frac{\left(\chi+q^3+2q^2+2q\right)\eta_1^2 y^3 E^2}{4\left(\chi+q\right)\left(\chi+q+q^2\right)\left(\chi+q+q^2+q^3\right)^2}$$

$$+\frac{\left[\chi^2+\chi\left(4q+4q^2+2q^3\right)+4q^2+8q^3+8q^4+4q^5+q^6\right]}{4\left(\chi+q\right)^2\left(\chi+q+q^2\right)^2\left(\chi+q+q^2+q^3\right)^2}\eta_1^2 E^2 y^2 - \frac{y^2E^2}{4\left(\chi+q+q^2\right)^2}$$

$$-\left[\frac{\chi^4+\chi^3\left(4q+q^2\right)+\chi^2\left(2+6q^2+3q^3\right)+\chi\left(8q+2q^2+6q^3+3q^4\right)+12q^2+12q^3+9q^4+3q^5}{24\left(\chi+q\right)^3\left(\chi+q+q^2\right)\left(\chi+q+q^2+q^3\right)^2}\right]\eta_1^4 Ey^2$$

$$+\left[\frac{\begin{array}{l}\chi^5+\chi^4\left(6q+3q^2+q^3\right)+\chi^3\left(2+14q^2+13q^3+6q^4+q^5\right)+\chi^2\left(12q+6q^2+20q^3+21q^4+12q^5+3q^6\right)\\ +\chi\left(28q^2+32q^3+33q^4+23q^5+12q^6+3q^7\right)+24q^3+48q^4+54q^5+36q^6+15q^7+3q^8\end{array}}{12\left(\chi+q\right)^4\left(\chi+q+q^2\right)^2\left(\chi+q+q^2+q^3\right)^2}\right]\eta_1^4 yE$$

$$-\frac{\left(\chi+q^2+2q\right)Ey\eta_1^2}{2\left(\chi+q\right)^2\left(\chi+q+q^2\right)^2}.$$

Now, with a constraining assumption, it can be written that $\eta_1 = \eta$, here $0 \le \eta \le 2$. Therefore, by applying triangle inequality, we obtain:

$$\left|a_4^2 - a_3^2\right| \le \Big(\frac{(2-\eta)^2(4-\eta^2)^2}{16(\chi+q+q^2+q^3)^2}|y|^4 + \frac{(\chi+2q+2q^2+q^3)(\eta^2-2\eta)(4-\eta^2)^2}{4(\chi+q)(\chi+q+q^2)(\chi+q+q^2+q^3)^2}|y|^3$$

$$+\Bigg[\frac{\eta^3(\eta-2)(4-\eta^2)R_1(\chi)}{24(\chi+q)^3(\chi+q+q^2)(\chi+q+q^2+q^3)^2} + \frac{R_2(\chi)\eta^2(4-\eta^2)^2}{4(\chi+q)^2(\chi+q+q^2)^2(\chi+q+q^2+q^3)^2}$$

$$+\frac{\eta(4-\eta^2)^2}{4(\chi+q+q^2+q^3)^2} - \frac{(\chi^2+2\chi(q+q^2-q^3)+q^2+2q^3-q^4-2q^5-q^6)(4-\eta^2)^2}{4(\chi+q+q^2)^2(\chi+q+q^2+q^3)^2}\Bigg]|y|^2$$

$$+\Bigg[\frac{(\chi+2q+2q^2+q^3)\eta(4-\eta^2)^2}{2(\chi+q)(\chi+q+q^2)(\chi+q+q^2+q^3)^2} + \frac{R_3(\chi)\eta^4(4-\eta^2)}{12(\chi+q)^4(\chi+q+q^2)^2(\chi+q+q^2+q^3)^2}$$

$$+\frac{(\chi+q^2+2q)(4-\eta^2)\eta^2}{2(\chi+q)^2(\chi+q+q^2)^2}\Bigg]|y| + \left|R_4(\chi)\eta^6 - R_5(\chi)\eta^4\right|$$

$$+\frac{(4-\eta^2)^2}{4(\chi+q+q^2+q^3)^2} + \frac{R_1(\chi)\eta^3(4-\eta^2)}{12(\chi+q)^3(\chi+q+q^2)(\chi+q+q^2+q^3)^2}$$

$$=\psi(\eta,|y|),$$

where,

$$R_1(\chi) = \chi^4 + \chi^3(4q+q^2) + \chi^2(2+6q^2+3q^3) + \chi(8q+2q^2+6q^3+3q^4) + 12q^2+12q^3+9q^4+3q^3,$$

$$R_2(\chi) = \chi^2 + \chi(4q+4q^2+2q^3) + 4q^2+8q^3+8q^4+4q^5+q^6,$$

$$R_3(\chi) = \chi^5 + \chi^4(6q+3q^2+q^3) + \chi^3(2+14q^2+13q^3+6q^4+q^5) + \chi^2(12q+6q^2+20q^3+21q^4+12q^5 +3q^6) + \chi(28q^2+32q^3+33q^4+23q^5+12q^6+3q^7) + 24q^3+48q^4+54q^5+36q^6+15q^7+3q^8$$

And

$$R_4(\chi) = \frac{\begin{bmatrix} \chi^8 + \chi^7(2q^2+8q) + \chi^6(q^4+14q^3+28q^2+4) + \chi^5(6q^5+42q^4+60q^3 \\ +8q^2+32q) + \chi^4(15q^6+78q^5+106q^4+68q^3+112q^2+4) + \\ \chi^3(24q^7+114q^6+180q^5+204q^4+216q^3+8q^2+32q) + \\ \chi^2(27q^8+126q^7+228q^6+300q^5+280q^4+80q^3+112q^2) + \\ \chi(18q^9+90q^8+192q^7+252q^6+384q^5+240q^4+192q^3) \\ +9q^{10}+54q^9+153q^8+288q^7+360q^6+288q^5+144q^4) \end{bmatrix}}{144(\chi+q)^6(\chi+q+q^2)^2(\chi+q+q^2+q^3)^2},$$

$$R_5(\chi) = \frac{\chi^2 + \chi(4q+2q^2) + 4q^2+4q^3+q^4}{4(\chi+q)^4(\chi+q+q^2)^2}.$$

For the maximum value of $\psi(\eta,|y|)$ on $[0,2]\times[0,1]$, firstly we suppose that the maximum is at an interior point $\psi(\eta_0,|y_0|)$ of $[0,2]\times[0,1]$, then on differentiation of $\psi(\eta,|y|)$ w. r. t. $|y|$ and then making it equal to zero gives $\eta=\eta_0=2$, which contradicts our assumption. Therefore, we consider only terminal points of $[0,2]\times[0,1]$ to get the maximum value of $\psi(\eta,|y|)$.

For $\eta=0$, we get:

$$\psi(0,|y|)=\frac{4}{(\chi+q+q^2+q^3)^2}|y|^4-\frac{4(\chi^2+2\chi(q+q^2-q^3)+q^2+2q^3-q^4-2q^5-q^6)}{(\chi+q+q^2)^2(\chi+q+q^2+q^3)^2}|y|^2+\frac{4}{(\chi+q+q^2+q^3)^2}.$$

Thus,

$$\psi(0,|y|)\le\frac{4}{(\chi+q+q^2)^2}.$$

For $\eta=2$, we get:

$$\psi(2,|y|)=|64R_4(\chi)-16R_5(\chi)|.$$

For $|y|=0$, we get:

$$\psi(\eta,0)=\left|R_4(\chi)\eta^6-R_5(\chi)\eta^4\right|+\frac{(4-\eta^2)^2}{4(\chi+q+q^2+q^3)^2}+\frac{R_1(\chi)\eta^3(4-\eta^2)}{12(\chi+q)^3(\chi+q+q^2)(\chi+q+q^2+q^3)^2}$$

$$\le\left|R_4(\chi)\eta^6-R_5(\chi)\eta^4\right|.$$

For $|y|=1$, we get:

$$\psi(\eta,1)=\frac{(2-\eta)^2(4-\eta^2)^2}{16(\chi+q+q^2+q^3)^2}+\frac{(\chi+2q+2q^2+q^3)(\eta^2-2\eta)(4-\eta^2)^2}{4(\chi+q)(\chi+q+q^2)(\chi+q+q^2+q^3)^2}$$
$$+\left[\frac{\eta^3(\eta-2)(4-\eta^2)R_1(\chi)}{24(\chi+q)^3(\chi+q+q^2)(\chi+q+q^2+q^3)^2}+\frac{R_2(\chi)\eta^2(4-\eta^2)^2}{4(\chi+q)^2(\chi+q+q^2)^2(\chi+q+q^2+q^3)^2}\right.$$
$$\left.+\frac{\eta(4-\eta^2)^2}{4(\chi+q+q^2+q^3)^2}-\frac{[\chi^2+2\chi(q+q^2-q^3)+q^2+2q^3-q^4-2q^5-q^6](4-\eta^2)^2}{4(\chi+q+q^2)^2(\chi+q+q^2+q^3)^2}\right]$$
$$+\left[\frac{(\chi+2q+2q^2+q^3)\eta(4-\eta^2)^2}{2(\chi+q)(\chi+q+q^2)(\chi+q+q^2+q^3)^2}+\frac{R_3(\chi)\eta^4(4-\eta^2)}{12(\chi+q)^4(\chi+q+q^2)^2(\chi+q+q^2+q^3)^2}\right.$$
$$\left.+\frac{(\chi+q^2+2q)(4-\eta^2)\eta^2}{2(\chi+q)^2(\chi+q+q^2)^2}\right]+\left|R_4(\chi)\eta^6-R_5(\chi)\eta^4\right|+\frac{(4-\eta^2)^2}{4(\chi+q+q^2+q^3)^2}$$
$$+\frac{R_1(\chi)\eta^3(4-\eta^2)}{12(\chi+q)^3(\chi+q+q^2)(\chi+q+q^2+q^3)^2},$$

which has maximum values $|64R_4(\chi)-16R_5(\chi)|$ for $\eta=2$ and $\dfrac{4}{(\chi+q+q^2)^2}$ for $\eta=0$.

Also, after simple calculations we obtain:

$$|64R_4(\chi)-16R_5(\chi)|=\frac{|R(\chi)|}{9(\chi+q)^6(\chi+q+q^2)^2(\chi+q+q^2+q^3)^2}.$$

Thus, the theorem is proved.

Remark 2.4 : For $q=1$, $\chi=0$, the above theorem coincides with the result $|T_2(3)|\le 7$ obtained by Thomas et al. [2] and for $q=1$, $\chi=0$, the above result gives the bound $|T_2(3)|\le 4/9$ which coincides with the results obtained by Sivasubramanian et al. [9].For $q=1$,theorem 2.3 yields the bound confirmed by Radhika et al. [1].

Theorem 2.5: Let ζ be the function given by equation (1.1) belongsto the class $N(\chi),(0\le\chi\le 1;\chi\ne\chi_0)$. Then,

$$|T_3(2)|=\left|\begin{pmatrix} a_2 & a_3 & a_4\\ a_3 & a_2 & a_3\\ a_4 & a_3 & a_2\end{pmatrix}\right|\le\begin{cases}\max\left\{|X_1(\chi)X_2(\chi)|,\dfrac{8|X_1(\chi)|}{(\chi+q+q^2)^2}\right\}; if\,\chi\ne\chi_0\\ \max\left\{|X_2(\chi)X_3(\chi)|,\dfrac{8|X_3(\chi)|}{(\beta+q+q^2)^2}\right\}; if\,\chi=\chi_0\end{cases},$$

where χ_0 is the positive root of the polynomial,

$$4\chi^4+\chi^3\left(16q+10q^2+6q^3\right)+\chi^2\left(6q^5+24q^4+30q^3+24q^2-4\right)+$$
$$\chi\left(12q^6+30q^5+30q^4+12q^3-4q^2-16q\right)+6q^7+12q^6+6q^5-12q^4-24q^3-24q^2=0,$$

and $$X_1(\chi)=\frac{\begin{bmatrix}4\chi^4+\chi^3\left(16q+10q^2+6q^3\right)+\chi^2\left(6q^5+24q^4+30q^3+24q^2-4\right)+\\ \chi\left(12q^6+30q^5+30q^4+12q^3-4q^2-16q\right)+6q^7+12q^6+6q^5-12q^4-24q^3-24q^2\end{bmatrix}}{3(\chi+q)^3(\chi+q+q^2)(\chi+q+q^2+q^3)}$$

$$X_2(\chi)=\frac{\begin{Bmatrix}4[4\chi^5+\chi^4\left(3q^3+11q^2+20q\right)+\chi^3\left(6q^5+22q^4+44q^3+40q^2-4\right)\\ +\chi^2\left(3q^7+21q^6+48q^5+66q^4+36q^3-14q^2-20q\right)+\\ \chi\left(6q^8+24q^7+42q^6+36q^5-10q^4-38q^3-28q^2\right)\\ +3q^9+9q^8+9q^7-9q^6-30q^5-24q^4-12q^3]\end{Bmatrix}}{3(\chi+q)^4(\chi+q+q^2)^2(\chi+q+q^2+q^3)},$$

$$X_3(\chi)=\frac{\begin{Bmatrix}8\chi^4+\chi^3\left(6q^3+14q^2+32q\right)+\chi^2\left(6q^5+24q^4+42q^3+48q^2+4\right)+\\ \chi\left(12q^6+30q^5+42q^4+36q^3+4q^2+16q\right)+6q^7+12q^6+18q^5+24q^4+24q^3+24q^2\right)\end{Bmatrix}}{3(\chi+q)^3(\chi+q+q^2)(\chi+q+q^2+q^3)}.$$

Proof: We can note,

$$|T_3(2)|=\left|a_2^3-2a_2a_3^2+2a_3^2a_4-a_2a_4\right|$$
$$=\left|(a_2-a_4)(a_2^2-2a_3^2+a_2a_4)\right|.$$

By applying similar procedure used in Theorem 2.3, we can get:

$|a_2-a_4|\le|X_1(\chi)|$ for $\chi\ne\chi_0$,

where,

$$X_1(\chi)=\frac{\begin{bmatrix}4\chi^4+\chi^3\left(16q+10q^2+6q^3\right)+\chi^2\left(6q^5+24q^4+30q^3+24q^2-4\right)+\\ \chi\left(12q^6+30q^5+30q^4+12q^3-4q^2-16q\right)+6q^7+12q^6+6q^5-12q^4-24q^3-24q^2\end{bmatrix}}{3(\chi+q)^3(\chi+q+q^2)(\chi+q+q^2+q^3)},$$

and χ_0 is the positive root of the polynomial,

$$4\chi^4+\chi^3\left(16q+10q^2+6q^3\right)+\chi^2\left(6q^5+24q^4+30q^3+24q^2-4\right)+$$
$$\chi\left(12q^6+30q^5+30q^4+12q^3-4q^2-16q\right)+6q^7+12q^6+6q^5-12q^4-24q^3-24q^2=0.$$

Now, using equations (2.2), (2.3) and (2.8) we get:

$$\left|a_2^2-2a_3^2+a_2a_4\right|=\left|\frac{\eta_1^2}{(\chi+q)^2}-\frac{2\eta_2^2}{(\chi+q+q^2)^2}-\frac{(\chi-1)^2(\chi+2q)^2\eta_1^4}{2(\chi+q)^4(\chi+q+q^2)^2}+\right.$$
$$\frac{\left[\chi^3+\chi^2(2q^3+3q-1)+\chi(3q^4-q^3+2q^2-3q)-2q^2-q^3-3q^4\right]\eta_1^2\eta_2}{(\chi+q)^2(\chi+q+q^2)^2(\chi+q+q^2+q^3)}$$
$$+\frac{\eta_1\eta_3}{(\chi+q)(\chi+q+q^2+q^3)}$$

$$\left.+\frac{[2\chi^4+\chi^3(-3+8q+2q^2)+\chi^2(1-12q+6q^2+3q^3)+\chi(4q-14q^2-3q^3)+6q^2]\eta_1^4}{6(\chi+q)^4(\chi+q+q^2)(\chi+q+q^2+q^3)}\right|$$

By the application of Lemma 1.1 and then by triangle inequality, with the assumption of $\eta_1=\eta$, where $0\le\eta\le2$, we obtain:

$$\left|a_2^2-2a_3^2+a_2a_3\right|\le\left[\frac{\left(\eta^2-2\eta\right)}{4(\chi+q)(\chi+q+q^2+q^3)}+\frac{(4-\eta^2)}{2(\chi+q+q^2)^2}\right](4-\eta^2)|y|^2$$
$$+\frac{[\chi^2+\chi(3q+q^2+q^3)+2q^2+2q^3+3q^4+q^5]\eta^2(4-\eta^2)}{2(\chi+q)^2(\chi+q+q^2)^2(\chi+q+q^2+q^3)}|y|$$
$$+\left|\frac{\eta^2}{(\chi+q)^2}-\frac{\begin{bmatrix}-\chi^5-\chi^4(5q+2q^2)-\chi^3(-4+10q^2+8q^3+q^4)\\-\chi^2(3q^5+12q^4+6q^3-14q^2-20q)-\chi(3q^6-25q^4-38q^3-28q^2)\\+12q^3+24q^4+33q^5+18q^6+3q^7\end{bmatrix}}{12(\chi+q)^4(\chi+q+q^2)^2(\chi+q+q^2+q^3)}\eta^4\right|$$
$$+\frac{\eta(4-\eta^2)}{2(\chi+q)(\chi+q+q^2+q^3)}$$
$$=\Omega(\eta,|y|)$$

Now for the highest value of $\Omega(\eta,|y|)$ on $[0,2]\times[0,1]$, we firstlysuppose that the maximum is at an interior point $\Omega(\eta_0,y_0)$. Differentiation of $\Omega(\eta,|y|)$ with respect to $|y|$ and then equating it to zero gives $\eta=\eta_0=2$, which contradicts our assumption. Therefore, the maximum is at terminal points of $[0,2]\times[0,1]$.

If $\eta=0$, then we get:

$$\Omega(0,|y|)=\frac{8}{(\chi+q+q^2)^2}|y|^2\le\frac{8}{(\chi+q+q^2)^2}.$$

For $\eta=2$, we get:

$$\Omega(2,|y|)=4\frac{\begin{bmatrix}4\chi^5+\chi^4(3q^3+11q^2+20q)+\chi^3(6q^5+22q^4+44q^3+40q^2-4)\\+\chi^2(3q^7+21q^6+48q^5+66q^4+36q^3-14q^2-20q)+\\\chi(6q^8+24q^7+42q^6+36q^5-10q^4-38q^3-28q^2)+\\3q^9+9q^8+9q^7-9q^6-30q^5-24q^4-12q^3\end{bmatrix}}{3(\chi+q)^4(\chi+q+q^2)^2(\chi+q+q^2+q^3)}$$
$$=\mathrm{X}_2(\chi).$$

For $|y|=0$, we get:

$$\Omega(\eta,0)=\left|\frac{\eta^2}{(\chi+q)^2}-\frac{\begin{bmatrix}-\chi^5-\chi^4(5q+2q^2)-\chi^3(-4+10q^2+8q^3+q^4)\\ -\chi^2(3q^5+12q^4+6q^3-14q^2-20q)-\chi(3q^6-25q^4-38q^3-28q^2)\\ +12q^3+24q^4+33q^5+18q^6+3q^7\end{bmatrix}}{12(\chi+q)^4(\chi+q+q^2)^2(\chi+q+q^2+q^3)}\eta^4\right|$$

$$+\frac{\eta(4-\eta^2)}{2(\chi+q)(\chi+q+q^2+q^3)},$$

which has maximum value $X_2(\chi)$ gained at the terminal point $\eta=2$.

If $|y|=1$, then

$$\Omega(\eta,1)=\left|\frac{\eta^2}{(\chi+q)^2}-\frac{\begin{bmatrix}-\chi^5-\chi^4(5q+2q^2)-\chi^3(10q^2+8q^3+q^4-4)\\ -\chi^2(3q^5+12q^4+6q^3-14q^2-20q)-\\ \chi(3q^6-25q^4-38q^3-28q^2)+12q^3+24q^4+33q^5+18q^6+3q^7)\end{bmatrix}\eta^4}{12(\chi+q)^4(\chi+q+q^2)^2(\chi+q+q^2+q^3)}\right|$$

$$+\frac{(4-\eta^2)^2}{2(\chi+q+q^2)^2}+\frac{\eta^2(4-\eta^2)}{4(\chi+q)(\chi+q+q^2+q^3)}$$

$$+\frac{[\chi^2+\chi(3q+q^2+q^3)+2q^2+2q^3+3q^4+q^5]\eta^2(4-\eta^2)}{2(\chi+q)^2(\chi+q+q^2)^2(\chi+q+q^2+q^3)},$$

which has its maximum $\Omega(\eta,1)=\dfrac{8}{(\chi+q+q^2)^2}$ at $\eta=0$ and $\Omega(\eta,1)=X_2(\chi)$ at $\eta=2$.

Therefore, $|a_2^2-2a_3^2+a_2a_4|\le\max\left\{|X_2(\chi)|,\dfrac{8}{(\chi+q+q^2)^2}\right\}$.

Hence,

$$|T_3(2)|=\left|(a_2-a_4)(a_2^2-2a_3^2+a_2a_4)\right|\le\max\left\{|X_1(\chi)X_2(\chi)|,\frac{8|X_1(\chi)|}{(\chi+q+q^2)^2}\right\}.$$

For $\chi=\chi_0$, we calculate $|a_2-a_4|$ as follows,

$$|a_2-a_4|=\left|\frac{\eta_1}{\chi+q}-\left(\frac{\eta_3}{(\chi+q+q^2+q^3)}-\frac{(\chi-1)(\chi+2q+q^2)\eta_1\eta_2}{(\chi+q)(\chi+q+q^2)(\chi+q+q^2+q^3)}\right)+\right.$$

$$\left.+\frac{[2\chi^4+\chi^3(2q^2+8q-3)+\chi^2(3q^3+6q^2-12q+1)+\chi(-3q^3-14q^2+4q)+6q^2]\eta_1^3}{6(\chi+q)^3(\chi+q+q^2)(\chi+q+q^2+q^3)}\right|.$$

Since, we know that each $|\eta_i|\le 2$, therefore by applying triangle inequality we get:

$$|a_2-a_4|\le|X_3(\chi)|.$$

Hence,

$$|T_3(2)| = |(a_2 - a_4)(a_2^2 - 2a_3^2 + a_2 a_4)| \le \max\left\{|X_2(\chi)X_3(\chi)|, \frac{8|X_3(\chi)|}{(\chi+q+q^2)^2}\right\}.$$

Hence the theorem is proved.

Remark 2.6 : For $q=1, \chi=0$, Theorem 2.5 gives the result $|T_3(2)| \le 8$ obtained by Thomas et al.[2] for the class of starlike functions S^* and for $q=1, \chi=1$ gives the bound $|T_3(2)| \le 4/9$ confirmed by Sivasubramanian et al. [9]. For $q=1, 0<\chi<1$, Theorem 2.5 coincides with the result obtained by Sivasubramanian et al. [1].

Theorem 2.7: Let ζ given by equation (1.1), belongsto the class $N(\chi), 0 \le \chi \le 1$. Then,

$$|T_3(1)| = \left|\begin{pmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{pmatrix}\right| \le \max\left\{1 + \frac{4}{(\chi+q+q^2)^2}, |X_4(\chi)|\right\},$$

where,

$$X_4(\chi) = \frac{\begin{bmatrix} \chi^6 + \chi^5(2q^2+6q) + \chi^4(q^4+10q^3+15q^2-8) \\ +\chi^3(4q^5+20q^4+20q^3-16q^2-32q) + \chi^2(6q^6+20q^5+7q^4-48q^3 \\ -48q^2+12) + \chi(4q^7+10q^6-10q^5-48q^4-32q^3+24q^2+32q) + q^8 \\ +2q^7-7q^6-16q^5+4q^4+32q^3+16q^2 \end{bmatrix}}{(\chi+q)^4(\chi+q+q^2)^2}.$$

Proof: First of all, we write determinant as follows,

$$T_3(1) = 1 + 2a_2^2(a_3 - 1) - a_3^2$$

Now by using equations (2.6) and (2.7) and with the use of Lemma 1.1, we get:

$$T_3(1) = 1 + \frac{2\eta_1^2}{(\chi+q)^2}\left(\frac{\eta_2}{(\chi+q+q^2)} - \frac{(\chi-1)(\chi+2q)}{2(\chi+q)^2(\chi+q+q^2)}\eta_1^2 - 1\right)$$

$$-\frac{\eta_2^2}{(\chi+q+q^2)^2} - \frac{(\chi-1)^2(\chi+2q)^2}{4(\chi+q)^4(\chi+q+q^2)^2}\eta_1^4 + \frac{(\chi-1)(\chi+2q)\eta_1^2\eta_2}{(\chi+q)^2(\chi+q+q^2)^2}.$$

Or,

$$T_3(1) = 1 + \frac{\eta_1^2(\eta_1^2 + Ey)}{(\chi+q)^2(\chi+q+q^2)} - \frac{(\chi-1)(\chi+2q)\eta_1^4}{(\chi+q)^4(\chi+q+q^2)} - \frac{2\eta_1^2}{(\chi+q)^2}$$

$$-\frac{\left[\eta_1^4 + E^2y^2 + 2\eta_1^2 Ey\right]}{4(\chi+q+q^2)^2} - \frac{(\chi-1)^2(\chi+2q)^2\eta_1^4}{4(\chi+q)^4(\chi+q+q^2)^2} + \frac{(\chi-1)(\chi+2q)\eta_1^2(\eta_1^2+Ey)}{2(\chi+q)^2(\chi+q+q^2)^2}$$

$$=1+\left[\frac{3\chi^2+\chi(6q^2+8q)+3q^4+8q^3+4q^2}{4(\chi+q)^4(\chi+q+q^2)^2}\right]\eta_1^4$$

$$+\frac{(\chi+q^2)\eta_1^2 Ey}{2(\chi+q)^2(\chi+q+q^2)^2}-\frac{E^2y^2}{4(\chi+q+q^2)^2}.$$

By generalising, we take $0\le\eta_1=\eta\le 2,$ and substitute this in the equation written above and then by the use of triangle inequality,

$$|T_3(1)|\le\frac{(4-\eta^2)^2}{4(\chi+q+q^2)^2}|y|^2+\frac{(\chi+q^2)(4-\eta^2)\eta^2}{2(\chi+q)^2(\chi+q+q^2)^2}|y|$$

$$+\left[1+\frac{\left[\left|\left(3\chi^2+\chi\left(6q^2+8q\right)+4q^2+8q^3+3q^4\right)\eta^2-8(\chi+q)^2(\chi+q+q^2)^2\right|\eta^2\right]}{4(\chi+q)^4(\chi+q+q^2)^2}\right],$$

or,

$$|T_3(1)|\le\frac{(4-\eta^2)^2}{4(\chi+q+q^2)^2}+\frac{(\chi+q^2)\eta^2(4-\eta^2)}{2(\chi+q)^2(\chi+q+q^2)^2}$$

$$+\left[1+\frac{\left[\left|\left(3\chi^2+\chi\left(6q^2+8q\right)+4q^2+8q^3+3q^4\right)\eta^2-8(\chi+q)^2(\chi+q+q^2)^2\right|\eta^2\right]}{4(\chi+q)^4(\chi+q+q^2)^2}\right]$$

$$=\varphi(\eta,\chi).$$

By differentiation of $\varphi(\eta,\chi)$ with respect to η we get:

$$\frac{\partial(\varphi(\eta,\chi)}{\partial\eta}=\frac{\eta\left[\begin{array}{l}\eta^2[\chi^4+\chi^3(4q-2)+\chi^2(4q^2-4q+3)+\chi(4q^2+8q)\\+2q^4+8q^3+4q^2]+\chi^3(8q^2+4)+\chi^2(4q^4+24q^3+4q^2+8q)+\\\chi(24q^4+8q^3+4q^2)+4q^6+8q^5+4q^4\end{array}\right]}{\left(\chi+q\right)^4(\chi+q+q^2)^2}.$$

By setting $\dfrac{\partial(\varphi(\eta,\chi))}{\partial\eta}=0$ gives either $\eta=0$ or

$$\eta^2=\frac{-\chi^3(8q^2+4)-\chi^2(4q^4+24q^3+4q^2+8q)-\chi(24q^4+8q^3+4q^2)-4q^6-8q^5-4q^4}{\chi^4+\chi^3(4q-2)+\chi^2(4q^2-4q+3)+\chi(4q^2+8q)+2q^4+8q^3+4q^2}.$$

But the above value is negative for $0\le\chi\le1, q\in N(q\ne1)$, so the highest is gained at the terminal points $\eta_1=\eta\in[0,2]$.

For $\eta_1=0$, we get:

$$|T_3(1)|=1+\frac{4|y|^2}{(\chi+q+q^2)^2}\le1+\frac{4}{(\chi+q+q^2)^2}.$$

For $\eta_1 = 2$, we get:

$$|T_3(1)| \le \left| 1 + \frac{4[3\chi^2 + \chi(6q^2 + 8q) + 4q^2 + 8q^3 + 3q^4]}{(\chi + q)^4 (\chi + q + q^2)^2} - \frac{8}{(\chi + q)^2} \right|$$

$$\le |X_4(\chi)|.$$

Hence the theorem is proved.

Remark 2.8: For $\chi = 0, q = 1$, theorem2.7 yields $|T_3(1)| \le 8$ which coincides with the result obtained by Thomas et al. [2], and for $\chi = 1, q = 1$, the above theorem yields $|T_3(1)| \le 13/9$ which coincides with that of Radhika et al. [9]. For $q = 1$, the above theorem becomes same as confirmed by Radhika et. al. [1].

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