On $-Closed Sets in Bi-ˇCech Closure Spaces

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Abstract

In this article, the idea of a $-closed set in a bi -ˇ Cech closure space is introduced, and some characterizations and features are examined. Additionally, the idea of ${}_{\$}C_0$ bi ˇ Cech spaces and ${}_{\$}C_1$ bi-ˇCech spaces are introduced, and their fundamental features are researched.

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1. Introduction

Cech closure spaces were introduced by ˇCech [3]. Numerous authors have since researched them [2,4,5,6,7,8,11,12,13,14]. In Cech's method, the operator satisfies the Kuratowski axiom idempotent requirement. This requirement does not have to be true for each set C of M. The operator becomes a topological closure operator when both of these conditions are satisfied. So, a topological space is a generalization of the idea of closure space. The idea of a gs-closed set was developed by Arya [1] et al. to examine various topological features. Hammer deserves credit for a proper description of closure functions; see, for instance, [10] and Gnilka [8]. The idea of bi -ˇ Cech -$ closed sets and some of its attributes are covered in this research.

2. Preliminaries

Definition: 2.1. [4] Two functions k_1 and k_2 from power set M to itself are called bi-ˇCech closure operators (simply bi-closure operators) for M if they satisfy the following properties.

(i) $k_1(\varphi) = \varphi$ and $k_2(\varphi) = \varphi$

(ii) $C \subset k_1(C)$ and $C \subset k_2(C)$ for any set $C \subseteq M$

(iii) $k_1(C \cup D)=k_1(C) \cup k_1(D)$ and $k_2(C \cup D)=k_2(C) \cup k_2(D)$ for any $C,D \subseteq M$

(M, k_1, k_2) is called bi- ˇ Cech closure space.

Example:2.1. Let M = {5, 6, 7} and define a closure operator k_1 on M by

$k_1(\{5\}) = \{5\}$, $k_1(\{6\}) = \{6, 7\}$, $k_1(\{7\}) = k_1(\{5, 7\})=\{5, 7\}$, $k_1(\{5, 6\}) = k_1(\{6, 7\}) = k_1(\{M\}) = M$, $k_1(\varphi) = \varphi$. Define a closure operator k_2 on M by $k_2(\{5\}) = \{5\}$, $k_2(\{6\}) = k_2(\{7\}) = k_2(\{6, 7\}) = \{6, 7\}$, $k_2(\{5, 6\}) = k_2(\{5, 7\}) = k_2(\{M\}) = M$, $k_2(\varphi) = \varphi$. Now, (M, k_1, k_2) is a bi- ˇ Cech closure space.

Definition: 2.2 [5] A subset C in a bi- ˇ Cech closure space (M, k_1, k_2) is said to be

1. k_i-semi open if $C \subseteq k_i(\text{int } k_i(C))$, i = 1, 2

2. k_i-semi closed if $\text{int } k_i(k_i(C)) \subseteq C$, i = 1, 2

The intersection of all ki - semi - closed subsets of M containing C is called the ki - semi - closure of C and is denoted by $k\text{-}scl_i(C)$

Definition: 2.3 [5] A subset C in bi -ˇ Cech closure space (M, k_1, k_2) is said to be a (k_1, k_2)-generalized semi closed set if $k\text{-}scl_2(C) \subseteq G$ whenever $C \subseteq G$ and G is k_1–open in (M, τ).

3. (k_1, k_2) -$ closed sets

Definition: 3.1 A subset C in bi -ˇ Cech closure space (M, k_1, k_2) is said to be a (k_1, k_2)-$ closed if $k\text{-}scl_2(C) \subseteq G$ whenever $C \subseteq G$ and G is k_{1}-gs open set in M.

Theorem: 3.1 If C and D are (k_1, k_2)-$ closed sets and so is a $C \cup D$.

Proof: Let C and D be the (k_1, k_2) $-closed sets. Let G be a k_1 gs-open set in M. Let $(C \cup D) \subseteq G$. Then $C \subseteq G$ and $D \subseteq G$. Then $k\text{-}scl_2(C) \subseteq G$ and $k\text{-}scl_2(D) \subseteq G$ Implies $(k\text{-}cl_2(C) \cup k\text{-}scl_2(D)) \subseteq G$. Hence $k\text{-}scl_2(C \cup D) \subseteq G$. Thus $C \cup D$ is a (k_1, k_2)-$ closed set.

Theorem: 3.2 If C is a (k_1, k_2)-$ closed set. Then $k\text{-}scl_2(C)$-C contains no non-empty

k_1-gs closed sets.

Proof: Let C be (k_1, k_2)-$ closed. Let G be k_1-gs closed contained in k-scl_2(C)-C.

Now, $G \subseteq$ k-scl_2(C) and $G \subseteq C^c$ ⟶ (1)

Now, $G \subseteq C^c$ then $C \subseteq G^c$. Since G is k_1-gs closed, G^c is k_1-gs open. Thus we have,

k-scl_2(C) $\subseteq G^c$.Consequently,

$G \subseteq$ [k-scl_2(C)]^c ⟶ (2)

From (1) and (2), $G \subseteq$ k-scl_2(C) $\cap$ [k-scl_2(C)]^c =Φ. Therefore G= Φ. Hence k-scl_2(C)-C contains no non-empty k_1-gs closed sets.

Theorem: 3.3 If C is a (k_1, k_2) -$ closed set, then k-scl_1(x) $\cap$ C ≠ φ holds for each x∈ k-scl_2(C)

Proof: Let C be a (k_1, k_2) -$ closed set. Suppose k-scl_1(x) ∩C=φ , for some x∈ k-scl_2(C),

We have $C \subseteq$ [k-scl_1(x)]^c. Now k-scl_1(x) is k_1-semi closed. Therefore [k-scl_1(x)]^c is k_1-semi open. Thus [k-scl_1(x)]^c is k_1-gs open. Since C is a (k_1, k_2) $-closed set, we have k-$scl_2$(C) $\subseteq$ [k-scl_1(x)]^c Implies k-scl_2(x) $\cap$ k-scl_1(x) = φ .Then x∉ k-scl_2(C) is a contradiction. Hence k-scl_2(x) $\cap$ C ≠ φ holds for each x∈ k-scl_2(C).

Theorem: 3.4 Let (M, k_1, k_2) be bi -ˇ Cech closure space. For each x in M, {x} is k_1-gs closed or $\{x\}^c$ is (k_1, k_2) -$ closed set.

Proof: Let (M, k_1, k_2) be bi-cech closure space. Suppose that {x} is not k_1-gs closed, $\{x\}^c$ is not k_1-gs open. Therefore, the only k_1-gs open set containing $\{x\}^c$ is M. Thus $\{x\}^c \subset$ M. Now, k-scl_2[$\{x\}^c$] $\subseteq$ k-scl_2(x)=M. Hence $\{x\}^c$ is a (k_1, k_2) -$ closed set.

Theorem: 3.5 Let C be a (k_1, k_2) -$ closed subset, and if C is k_1-gs open then C= k-scl_2(C).

Proof: Let C be a (k_1, k_2) -$ closed subset of a bi -ˇ Cech closure space (M, k_1, k_2) and let C be a k_1-gs open set. Then k-scl$_2$(C) ⊆ G, whenever C⊆ G and G is a k_1-gs open set in M. Since C is k_1-gs open and C ⊆ C, We have k-scl$_2$(C) ⊆ C but always, C ⊆ k-scl$_2$(C)Thus, C= k-scl$_2$(C).

Theorem: 3.6 Let C ⊆ Y⊆ M and suppose that C is (k_1, k_2) -$ closed in (M, k_1, k_2). Then C is (k_1, k_2)-$ closed relative to Y.

Proof: Let S be any k_1-gs open set in Y such that C ⊆ S. Then S=G ∩ Y for some G is k_1-gs open in M. Therefore C⊂ G ∩ Y implies C ⊆ G. Since C is a (k_1, k_2) -$ closed set in M, We have k-scl$_2$(C) ⊆ C. Hence Y ∩ k-scl$_2$(C) ⊆ Y ∩ G =S. Thus C is a (k_1, k_2)-$ closed set relative to Y.

4. $_\$C_0$ bi- ˇ Cech spaces and $_\$C_1$ bi- ˇ Cech spaces

Definition 4.1

A bi- ˇ Cech closure space (M,k_1,k_2) is said to be a $_\$\mathbf{C_0}$ bi- ˇ Cech space if for every (k_1, k_2)-$-open subset G of (M,k_1), $x \in G$ implies $k_2(\{x\}) \subseteq G$.

Example 4.1

Let M = {5, 6, 7} and define a closure operator k_1 on M by $k_1 (\{\varphi\}) = \varphi$, $k_1 (\{5\}) = \{5\}$,

$k_1 (\{6\}) = k_1 (\{7\}) = k_1 (\{6, 7\}) = \{6, 7\}$ and $k_1 (\{5, 6\}) = k_1 (\{5, 7\}) = k_1 (M) = M$. Define a closure operator k_2 on M by $k_2 (\{\varphi\}) = \varphi$, $k_2 (\{5\}) = \{5\}$, $k_2 (\{6\}) = \{6, 7\}$, $k_2 (\{7\}) = k_2 (\{5, 7\}) = \{5, 7\}$and $k_2 (\{5, 6\}) = k_2 (\{6, 7\}) = k_2 (M) = M$. Then (M, k_1, k_2) is a $_\$\mathbf{C_0}$ bi- ˇ Cech space.

Theorem 4.1

A bi- ˇCech space (M,k_1,k_2) is a $_\$\mathbf{C_0}$ bi- ˇ Cech space if and only if for every (k_1, k_2)-$-closed subset H of (M,k_1) such that $x \notin H$, $k_2(\{x\}) \cap H = \phi$

Proof

Let H be a (k_1, k_2)-$ -closed subset of (M,k_1) and let $x \notin H$, since $x \in M-H$ and M-H is a (k_1, k_2)-$ -open subset of (M,k_1), $k_2(\{x\}) \subseteq M-H$. Consequently, $k_2(\{x\}) \cap H = \phi$.

Conversely, let G be a \$-open subset of (M,k_1) and let $x\in G$.Since M-U is a \$-closed subset of (M,k_1), and $x\notin M-G$, $k_2(\{x\})\cap(M-G)=\phi$. Consequently $k_2(\{M\})\subseteq U$. Hence (M,k_1,k_2) is $_\$$ **C_0** bi -ˇ Cech space.

Definition 4.2

A bi -ˇ Cech closure space (M,k_1,k_2) is said $_\$$ **C_1** bi -ˇ Cech space if for each x, $y\in M$ such that $k_1(\{x\})\neq k_2(\{y\})$,there exist disjoint (k_1, k_2)-\$ -open subset G of (M,k_2) and (k_1, k_2)-\$ -open subset H of (M, k_1) such that $k_1(\{x\})\subseteq G$ and $k_2(\{y\})\subseteq H$.

Example 4.2

Let M = {5, 6} and define a closure operator k_1 on M by $k_1(\{\varphi\}) = \varphi$ and $k_1(\{5\}) = k_1(M) = M$. Define a closure operator k_2 on M by $k_2(\{\varphi\}) = \varphi$ and $k_2(\{6\}) = k_2(M) = M$. Then (M, k_1, k_2) is a $_\$$ **C_1** bi- ˇ Cech space.

Theorem 4.2

Every $_\$$ **C_1** bi ˇ Cech space is a $_\$$ **C_0** bi -ˇ Cech space

Proof

Let (M,k_1,k_2) be a $_\$$ **C_1** bi ˇ Cech space .Let G be a (k_1, k_2)-\$ -open subset of (M,k_1) and let $x\in G$. If $y\notin G$, then $k_2(\{x\})\neq k_1(\{y\})$ because $x \notin k_1(\{y\})$.Then there exist a (k_1, k_2)-\$ -open subset H_y of (M,k_2) such that $k_1(\{y\}) \subseteq H_y$ and $x \notin V_y$, which implies $y\notin k_2(\{x\})$.Consequently $k_2(\{y\}) \subseteq G$. Hence (M,k_1,k_2) is a $_\$$ **C_0** bi ˇ Cech space.

The converse need not be true as seen from the following example

Example 4.3

Let M = {5, 6} and define a closure operator k_1 on M by $k_1(\{\varphi\}) = \varphi$ and $k_1(\{5\}) = k_1(M) = M$. Define a closure operator k_2 on M by $k_2(\{\varphi\}) = \varphi$, $k_2(\{5\}) = \{5\}$ and $k_2(\{b\}) = k_2(M) = M$. Then $(M, k_1\ k_2,)$ is a $_\$$ **C_0** bi- ˇ Cech space but it is not a $_\$$ **C_1** bi- ˇ Cech space.

Theorem 4.3

A bi ˇ Cech closure space (M,k_1,k_2) is a $_{\$}\mathbf{C_1}$ bi ˇ Cech space if and only if for every pair of points x, y of (M,k_1,k_2) such that $k_1(\{x\}) \neq k_2(\{y\})$ there exists a \$-open subset G of (M,k_2) and (k_1, k_2)-\$ -open subset H of (M,k_2) such that $x \subseteq H$, $y \subseteq G$ and $H \cap G = \phi$

Proof

Suppose that (M,k_1,k_2) is a $_{\$}\mathbf{C_1}$ bi ˇ Cech space. Let x,y be points of (M,k_1,k_2) such that $k_1(\{x\}) \neq k_2(\{y\})$.There exists a (k_1, k_2)-\$-open subset G of (M,k_1) and (k_1, k_2)-\$ -open subset H of (M,k_2) such that $x \in k_1(\{x\}) \subseteq H$ and $y \in k_2(\{y\}) \subseteq G$.

Conversely, suppose that there exist a (k_1, k_2)-\$-open subset G of (M,k_1) and open subset H of (M,k_2) such that $x \subseteq H$ and $y \subseteq G$ and $G \cap H = \phi$ .Since every $_{\$}\mathbf{C_1}$ bi -ˇ Cech space is $_{\$}\mathbf{C_0}$ bi-Cech space, $k_1(\{x\}) \subseteq H$ and $k_2(\{y\}) \subseteq G$.

Theorem 4.4

Let $\{ (M,k_i^1,k_i^2) : i \in I\}$ be a family of bi -ˇ Cech closure spaces. If $\prod_{i \in I} (M,k_i^1,k_i^2)$ is an $_{\$}\mathbf{C_0}$ bi ˇ Cech space, then (M,k_i^1,k_i^2) is an $_{\$}\mathbf{C_0}$ bi ˇ Cech space for each $i \in I$.

Proof

Suppose that $\prod_{i \in I} (M,k_i^1,k_i^2)$ is an $_{\$}\mathbf{C_0}$ bi ˇ Cech space.Let $j \in I$ and let G be an (k_1, k_2)-\$ -open subset of (M_j,k_j^1) such that $x_j \in G$. Then $G \times \prod_{\substack{i \neq j \\ i \in i}} M_i$ is an (k_1, k_2)-\$ -open subset of $\prod_{i \in I} (M,k_i^1)$ such that $(M_i)_{i \in I} \in G \times \prod_{\substack{i \neq j \\ i \in i}} M_i$ .Since $\prod_{i \in I} (M,k_i^1,k_i^2)$ is an $_{\$}\mathbf{C_0}$ bi ˇ Cech space, $\prod_{i \in I} k_i^2 \prod_{i \in I} (\{(x_i)_{i \in I}\}) \subseteq G \times \prod_{\substack{i \neq j \\ i \in i}} M_i$. Consequently, $k_j^2\{x_j\} \subseteq G$. Hence (M,k_i^1,k_i^2) is an $_{\$}\mathbf{C_0}$ bi ˇ Cech space.

Theorem 4.5

Let $\{(M, k_i^1,k_i^2) : i \in I \}$ be a family of bi ˇ Cech closure spaces. If (M,k_i^1,k_i^2) is a $_{\$}\mathbf{C_1}$ bi ˇ Cech space for each $i \in I$, then $\prod_{i \in I} (M,k_i^1,k_i^2)$ is an $_{\$}\mathbf{C_1}$ bi ˇ Cech space.

Proof

Suppose that (M,k_i^1,k_i^2) is an ${}_{\$}\mathbf{C_1}$ bi -ˇ Cech space for each $i \in I$. Let $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$ be points of $\prod_{i\in I}(M_i)$ such that $\prod_{i\in I} k_i^2 \prod_{i\in I}(\{(x_i)_{i \in I}\}) \neq \prod_{i\in I} k_i^2 \prod_{i\in I}(\{(y_i)_{i \in I}\})$.There exists $j \in I$ such that $k_j^1\{x_j\} \neq k_j^2\{y_j\}$.Since (M,kj^1,k_j^2) is a ${}_{\$}\mathbf{C_1}$ bi ˇ Cech space, there exists an (k_1, k_2)-\$ -open subset G of (M_j,k_j^1) and an (k_1, k_2)-\$ -open subset H of (M_j,k_j^2) such that $G \cap H=\phi$, $k_j^1\{y_j\} \subseteq G$ and $k_j^1\{x_j\} \subseteq H$. Consequently $\prod_{i\in I} k_i^2 \prod_{i\in I}(\{(y_i)_{i \in I}\}) \subseteq G \times \prod_{\substack{i\neq j\\ i\in i}} M_i$ and

$\prod_{i\in I} k_i^1 \prod_{i\in I}(\{(x_i)_{i \in I}\}) \subseteq H \times \prod_{\substack{i\neq j\\ i\in i}} M_i$ such that $G \times \prod_{\substack{i\neq j\\ i\in i}} M_i$ is an \$-open subset of $\prod_{i\in I}(M_i,k_i^1)$, $H \times \prod_{\substack{i\neq j\\ i\in i}} M_i$ is an (k_1, k_2)-\$ -open subset of $\prod_{i\in I}(M_i,k_i^2)$ and $(G \times \prod_{\substack{i\neq j\\ i\in i}} M_i) \cap (H \times \prod_{\substack{i\neq j\\ i\in i}} M_i)= \phi$. Hence $\prod_{i\in I}(M_i,k_i^1,k_i^2)$ is an ${}_{\$}\mathbf{C_1}$ bi -ˇ Cech space.

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