Some Characterizations of Total PerfectDominating Sets in Graphs

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Abstract: A total perfect dominating set of a graph G, denoted by $\gamma_{tp}(G)$, is a subset S of its vertex set V(G) that satisfies two conditions: first, it is a total dominating set, meaning every vertex in the graph is either in S or adjacent to a vertex in S. Second, it is a perfect dominating set, meaning every vertex not in S has at least one neighbor in S. The total perfect domination number, $\gamma_{tp}(G)$, is the cardinality of the smallest total perfect dominating set in G. A $\gamma_{tp} - set$ of G is a total perfect dominating set of G that has cardinality equal to $\gamma_{tp}(G)$.

This research paper presents various characterizations of total dominating sets in graphs and determines the values or bounds of this parameter. Additionally, it delves into characterizing total perfect dominating sets in the join and corona of graphs and determines the corresponding values of the $\gamma_{tp}(G)$ parameter in these cases. The paper provides valuable insights into the properties and sizes of total perfect dominating sets in different graph structures.

Keywords: Domination, total domination, perfect domination, total perfect domination.

INTRODUCTION

Let $G = (V(G), E(G))$ be a graph and $v \in V(G)$. The open neighborhood of v is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$ And the closed neighborhood of v is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. If $S \subseteq V(G)$, then the open neighborhood of S is the set $N_G(S) = N(S) = \cup_{v \in S} N_G(v)$ And the closed neighborhood of S is the set $N_G[S] = N[S] = S \cup N(S)$.

A subset S of $V(G)$ is a dominating set of G if for every $v \in (G)\setminus S$, there exists $u \in S$ such that $uv \in E(G)$, that is, $N_G[S] = V(G)$. A subset S of $V(G)$ is a total dominating set of G if every vertex of G is adjacent to some vertex in S. It is a perfect dominating set of G if for each $v \in V(G)\setminus S$, is adjacent to exactly one vertex in S. The perfect domination number of G denoted by $\gamma(G)$ is the cardinality of the smallest perfect dominating set of G. A perfect dominating set of G with cardinality equal to $\gamma p(G)$ is called a γp-set of G.

Total domination was investigated in [1] and [2] where the bounds and some properties of this type of dominating set were characterized. On the other hand, the perfect dominating set was investigated in [2], [4], [5] and [6] where the bounds and some properties of the perfect dominating set were characterized.

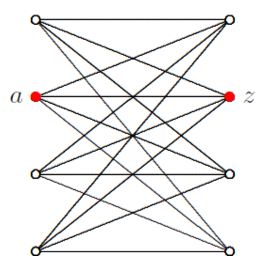
A subset S of $V(G)$ is a total perfect dominating set of G if S is both total and perfect dominating set of G. The total perfect domination number of G denoted by $\gamma_{tp}(G)$ is the cardinality of the smallest total perfect dominating set of G. A total perfect dominating set of G with cardinality equal to $\gamma_{tp}(G)$ is called a $\gamma_{tp} - set$ of G.

A total perfect dominating set (TPDS) in a graph is a fundamental concept in graph theory with diverse applications. In this section, we present key results and properties related to TPDS in graphs, shedding light on the significance of this concept in various contexts. The following findings contribute to our understanding of TPDS and its implications in solving real-world problems.

RESULTS

Theorem 1. Let $m \geq 1$ and $n \geq 1$. A nonempty subset S of $(K_{m,n})$ is a total perfect dominating set of $K_{m,n}$ if and only if $S = S_1 \cup S_2$ where S_1 and S_2 are singleton subsets of $V(\overline{\quad}_m)$ and $V(\overline{\quad}_n)$, respectively.

Proof: Assume that S is a total perfect dominating set of $K_{m,n}$. Since $K_{m,n} = \overline{K_m} + V(\overline{\quad}_n)$, it follows that $S \subsetneq V(\overline{K_m})$ and $S \subsetneq V(\overline{K_n})$. Hence, $S = S_1 \cup S_2$ where $\phi = S_1 \subseteq V(\overline{\quad}_m)$ and $\phi = S_2 \subseteq V(\overline{\quad}_n)$. Suppose that $2 \leq |S_1| \leq m$. Then S is not a perfect dominating set of $K_{m,}$ which contradicts the assumption. Then, $|S_1| = 1$ or $|S_1| = 0$. Since $|S_1| \neq \phi$, $|S_1| = 1$. Similarly, $|S_2| = 1$. Therefore S_1 and S_2 are singleton subsets of $V(\overline{\quad}_m)$ and $V(\overline{K_n})$, respectively. For the converse, suppose that $S = S_1 \cup S_2$ where $|S_1| = |S_2| = 1$ and $S_1 \subseteq V(\overline{K_m})$ and $S_2 \subseteq V(\overline{K_n})$. Let $S = \{a, z\}$ where $a \in V(\overline{K_m})$ and $z \in V(\overline{K_n})$. Then S is a total domination set of $K_{m,n}$. Let $v \in V(\overline{K_m})$. Since $\overline{K_m}$ and $\overline{K_n}$ are empty graphs $N(v) \cap S = \{z\}$. Similarly, $(w) \cap S = \{a\}$ for all $w \in V(\overline{K_n})$. Therefore, S is a total perfect dominating set of $K_{m,n}$. ■



The graph of $K_{m,}$

Corollary 2. For a complete bipartite graph $K_{m,}$ where $m \geq 2$ and $n \geq 2$, $\gamma_{tp}(K_{m,n}) = 2$.

Proof: Let $m\geq 1$ and $n \geq 1$ and suppose $a \in V\ (\overline{\ \ }_m)$ and $z \in V\ (\overline{K}_n)$. Then $S = \{a, z\}$ is a total perfect dominating set of $K_{m,n}$ by Theorem 1. Since no singleton sets are total dominating sets, it follows that S in the total perfect dominating set with the smallest cardinality. Hence, S is a $\gamma_{tp} - set$ of $K_{m,n}$. Therefore, $\gamma_{tp}\left(K_{m,n}\right) = |S| = 2$. ■

Corollary 3. For a star $K_{1,}$ where $n\geq 1$, $\gamma_{tp}\left(K_{1,n}\right) = 2$.

Proof: Let $= V\left(\overline{K_1}\right) \cup$ where $|D|=1$ and $D \subseteq V\ (\overline{K}_n)$. By Theorem 1, S is a total perfect dominating set of $K_{m,n}$. By Corollary 2, $\gamma_{tp}\left(K_{1,}\right) = |S| = 2$. ■

The study of TPDS has proven beneficial in solving real-world problems across various domains. For instance, in wireless sensor networks, selecting an optimal TPDS can significantly reduce energy consumption while maintaining network connectivity. TPDSs have also found applications in social network analysis, where identifying influential individuals is crucial. Understanding these properties helps in devising efficient algorithms to find TPDSs and analyzing their impact on the overall graph structure.

The investigation of total perfect dominating sets in graphs yields valuable insights into their theoretical properties and practical applications. The results presented here contribute to the ongoing research on TPDS, guiding the development of algorithms and strategies for utilizing TPDSs in various graph-related problems. Further exploration of this concept holds promise for addressing complex challenges in diverse fields.

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