SQUARE HARMONIC MEAN LABELING OF SIMPLE GRAPHS

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Abstract:

If there is an injective function $h : V(G) \to \{1,2,\dots,q+1\}$ such that an induced edge function $h^* : E(G) \to \{1,2,\dots,q\}$ defined by $h^*(e = uv) = \left\lceil \frac{2h(u)^2 h(v)^2}{h(u)^2 + h(v)^2} \right\rceil$ or $\left\lfloor \frac{2h(u)^2 h(v)^2}{h(u)^2 + h(v)^2} \right\rfloor$ is bijective, then a graph $G = (V, E)$ with p vertices and q edges is called a square harmonic labeling**.** A graph which admits a square harmonic mean labeling is called a square harmonic mean graph. In this paper we prove that Path, Cycle, Comb, Star Graph, Crown Graph, Ladder Graph,Caterpillar, Olive Tree, Triangular Snake, Quadrilateral Snake and Hexagonal Snake Graph are square harmonic mean labeling of graphs.

Keywords: Graphs, Square harmonic mean graph.

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1. Introduction

All graphs considered here are simple, finite, connected and undirected graph. Graph labeling plays an important role in graph theory. The paper written by Leonhard Euler on the seven bridges of Konigsberg and published in 1936 is regarded as the first paper in the history of graph theory. It is used in many applications like coding theory, radio astronomy, X-ray crystallography and circuit design. A graph labelling is an assignment of integers to the vertices, edges or both of a graph with certain conditions. There is an enormous literature dealing with several kinds of labeling of graphs over the past three decades.We cite J. A. Gallian [1] for an extensive examination of graph labeling. We adhere to Harary's [2] conventions for all other terms and notations. Some more results on harmonic mean graphs have been examined by C. Jayasekaran, C. David Raj [3]. S. Somasundaram and R. Ponraj [4] are the ones who proposed the idea of mean labeling. S. Somasundaram, R. Ponraj and S.S. Sandhya [5] established the notion of harmonic mean labeling. The aforementioned studies served as our inspiration as we introduced square harmonic mean labeling and also investigate square harmonic mean labeling behaviour of some simple graphs.

Definition 1.1. If there is an injective function $h : V(G) \to \{1,2,\dots,q+1\}$ such that an induced edge function $h^* : E(G) \to \{1,2,\dots,q\}$ defined by $h^*(e = uv) = \left\lceil \frac{2h(u)^2 h(v)^2}{h(u)^2 + h(v)^2} \right\rceil$ or $\left\lfloor \frac{2h(u)^2 h(v)^2}{h(u)^2 + h(v)^2} \right\rfloor$ is bijective, then a graph $G = (V, E)$ with p vertices and q edges is called a **square harmonic labeling.** A graph which admits a square harmonic mean labeling is called a **square harmonic mean graph**.

Definition 1.2. Path refers to a walk where each of the vertices $u_0 u_1 \dots u_n$ are distinct. A path on n vertices is denoted by P_n.

Definition 1.3. Cycle of G refers to a closed path. The symbolC_n stands for a cycle on n vertices.

Definition 1.4. The graph $G = G_1 \cup G_2$formed by taking one copy of G_1 and $V(G_1)$ copies of G_2, where the i^{th} vertex of G_1 is next to every vertex in the i^{th} copy of G_2 is known as the **corona** $G_1 \odot G_2$ of two graphs G_1 and G_2.

Definition 1.5. Combrefers to the graph formed by connecting a single pendant edge to each path vertex.

Definition 1.6. A **Star Graph** is a complete bipartite graph $K_{1,n}$ is a tree with one internal node and n leaves.

Definition 1.7. Any cycle with a pendant edge attached at each vertex is called a **Crown Graph** and it is denoted by $C_n \odot K_1$.

Definition 1.8. The Cartesian product of a path on two vertices and another path on n vertices is called the **Ladder Graph**L_n, which has the form $P_2 \times P_n$.

Definition 1.9. A tree which yields a path when its pendant vertices are removed is called **Caterpillar.**

Definition 1.10. An **Olive tree** O_n is a collection of k paths joined in one of the end vertices, where the n^{th} path has n as its length.

Definition 1.11. A **Triangular Snake**$T_n$is obtained from a path $u_1, u_2, \dots, u_n$ by joining u_α and $u_{\alpha+1}$ to a new vertex v_i for $1 \leq \alpha \leq n-1$. That is every edge of a path is replaced by a triangle C_3.

Definition 1.12. A **Quadrilateral Snake**$Q_n$is a graph obtained from a path $u_1 u_2 \dots u_n$ by joining u_α and $u_{\alpha+1}$ to two new vertices v_α and w_α respectively, $1 \leq \alpha \leq n-1$ and then joining v_α and w_α. That is every edge of a path is replaced by a cycle C_4.

Definition 1.13. Hexagonal Snake Graph H_n has been defined as a connected graph in which all the blocks are isomorphic to the cycle C_6 and the block cut point graph is a path P_n, where P_n is path of minimum length that contains all the cut vertices of a hexagonal scale. That is every edge of path $v_1 v_2 \dots v_n$ of size n is replaced by a cycle C_6 and $d(v_i v_{i+1}) = 2$.

2. Main Results

Theorem 2.1. Path P_n admits a square harmonic mean graph.

Proof. Let $G = P_n$ be a path $u_1 u_2 \dots u_n$. Let $V(G) = \{u_\alpha : \alpha = 1, 2, \dots, n\}$ and $E(G) = \{u_\alpha u_{\alpha+1} : \alpha = 1,2,\dots, n-1\}$.Then $|V(G)| = n$ and $|E(G)| = n-1$.A function $h : V(G) \to \{1,2,\dots, q+1\}$is defined by $h(u_\alpha) = \alpha$, $\alpha = 1,2,\dots, n$. The corresponding induced edge label is$h^*(u_\alpha u_{\alpha+1}) = \alpha, \alpha = 1,2,\dots, n-1$. Thus h^* is bijective.Therefore, P_n admits a square harmonic mean graph.

Illustration 2.2. The image below displays a square harmonic mean labeling of P_6.

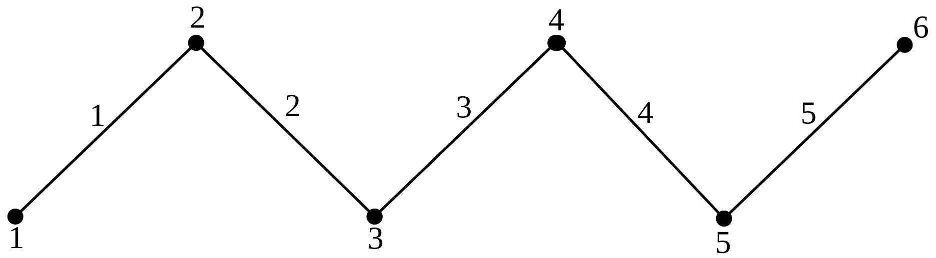

Figure I. P_6

Theorem 2.3. Cycle $C_n$admits a square harmonic mean graph.

Proof. Let$C_n$be the cycle of length n.Let the cycle be $u_1 u_2 \dots u_n u_1$.Let$V(C_n) = \{u_\alpha : \alpha = 1,2,\dots, n\}$ and $E(C_n) = \{u_\alpha u_{\alpha+1}, u_n u_1 : \alpha = 1,2,\dots, n-1\}$.Then$|V(C_n)| = n$ and $|E(C_n)| = n$. A function$h : V(C_n) \to \{1,2,\dots, q+1\}$is defined by$h(u_\alpha) = \alpha, \alpha = 1,2,\dots, n$**.** Moreover, the induced edge labels are$h^*(u_\alpha u_{\alpha+1}) = \alpha + 1$, $\alpha = 1,2,\dots, n-1$,$h^*(u_n u_1) = 1$**.** Thus h^* is bijective. Therefore,$C_n$admits a square harmonic mean graph

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Illustration 2.4. The image below displays asquare harmonic mean labeling of C_6.

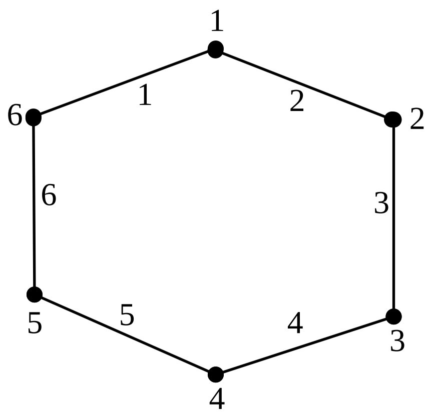

Figure II.C_6

Theorem 2.5. Comb $P_n \odot K_1$ admits a square harmonic mean graph.

Proof. Let us take $P_n = u_1u_2 \dots u_n$ and join a vertex u_α to corresponding pendant vertices v_α. Let G be comb with $V(G) = \{u_\alpha, v_\alpha : \alpha = 1,2,\dots,n\}$ and $E(G) = \{u_\alpha v_\alpha, u_n v_n, u_\alpha u_{\alpha+1} : \alpha = 1,2,\dots,n-1\}$. Then $|V(G)| = 2n$ and $|E(G)| = 2n-1$. A function $h : V(G) \to \{1,2,\dots,q+1\}$ is defined by $h(u_\alpha) = 2\alpha - 1$, $\alpha = 1,2,\dots,n$, $h(v_\alpha) = 2\alpha, \alpha = 1,2,\dots,n$. The corresponding induced edge labels are $h^*(u_\alpha u_{\alpha+1}) = 2\alpha, \alpha = 1,2,\dots,n-1$, $h^*(u_\alpha v_\alpha) = 2\alpha - 1$, $\alpha = 1,2,\dots,n$. Thus h^* is bijective. Therefore, G admits a square harmonic mean graph.

Illustration2.6. The image below displays asquare harmonic mean labeling of $P_5 \odot K_1$.

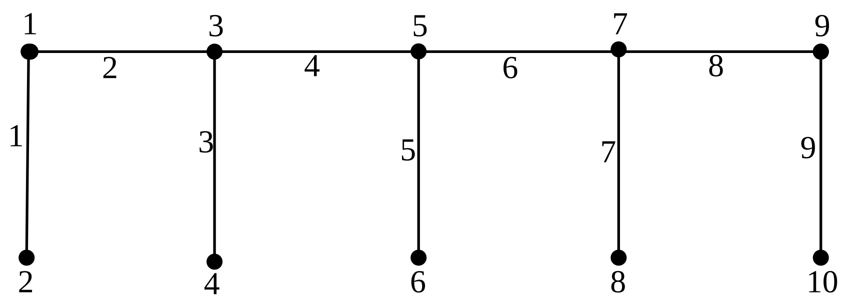

Figure III.$P_5 \odot K_1$

Theorem 2.7. Star Graph $K_{1,n}$ admits a square harmonic mean graph.

Proof. Let $G = K_{1,n}$ be a star graph. Let $V(G) = \{u, u_\alpha : \alpha = 1,2,\dots,n\}$ and $E(G) = \{uu_\alpha : \alpha = 1,2,\dots,n\}$. Then $|V(G)| = n+1$ and $|E(G)| = n$ edges. A function $h : V(G) \to \{1,2,\dots,q+1\}$ is defined by $h(u) = 1$, $h(u_\alpha) = \alpha + 1$, $\alpha = 1,2,\dots,n$. The corresponding induced edge label is $h^*(uu_\alpha) = \alpha$, $\alpha = 1,2,\dots,n$. Thus h^* is bijective. Therefore, star graph $K_{1,n}$ admits a square harmonic mean graph.

Illustration 2.8. The image below displays a square harmonic mean labeling of $K_{1,8}$.

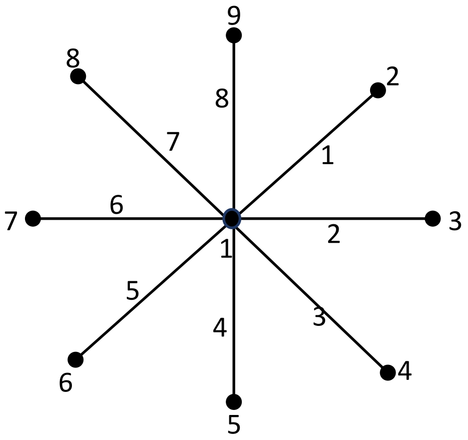

Figure IV.$K_{1,8}$

Theorem 2.9 Crown $C_n \odot K_1$ admits a square harmonic mean graph.

Proof. Let C_n be cycle $u_1u_2 \dots u_nu_1$ and v_α be the pendant vertices adjacent to $u_\alpha, \alpha = 1,2,\dots,n$.The resultant graph G is the crown $C_n \odot K_1$ with $V(G) = \{u_\alpha, v_\alpha : \alpha = 1,2,\dots,n\}$ and $E(G) = \{u_\alpha u_{\alpha+1}, u_nu_1, u_\alpha v_\alpha, v_nu_n : \alpha = 1,2,\dots,n-1\}$. Then $|V(G)| = 2n$ and $|E(G)| = 2n$. A function $h : V(G) \to \{1,2,\dots,q+1\}$is defined by $h(u_\alpha) = 2\alpha, \alpha = 1,2,\dots,n,$ $h(v_\alpha) = 2\alpha - 1,$ $\alpha = 1,2,\dots,n$. The corresponding induced edge labels are$h^*(u_1u_2) = 2,$ $h^*(u_\alpha u_{\alpha+1}) = 2\alpha + 1,$ $\alpha = 2,3,\dots,n-1,$ $h^*(u_nu_1) = 4,$ $h^*(v_1u_1) = 1,$ $h^*(v_2u_2) = 3,$ $h^*(v_\alpha u_\alpha) = 2\alpha,$ $\alpha = 3,4,\dots,n$. Thus h^* is bijective. Therefore,$C_n \odot K_1$ admits a square harmonic mean graph.

Illustration 2.10. The image below displays a square harmonic mean labeling of $C_6 \odot K_1$.

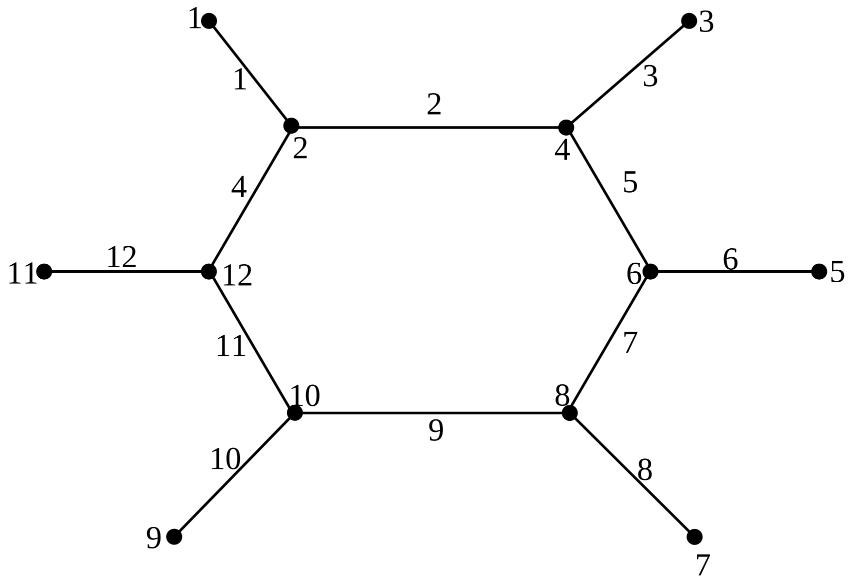

Figure V.$C_6 \odot K_1$

Theorem 2.11. Ladder L_n admits a square harmonic mean graph.

Proof. Let L_n be the ladder that joins the two paths $u_1u_2 \dots u_n$ and $v_1v_2 \dots v_n$ respectively.Let $V(L_n) = \{u_\alpha, v_\alpha : \alpha = 1,\dots,n\}$and $E(L_n) = \{u_\alpha u_{\alpha+1}, v_\alpha v_{\alpha+1}, u_\alpha v_\alpha, u_nv_n : \alpha = 1,\dots,n-1\}$ Then $|V(L_n)| = 2n$ and $|E(L_n)| = 3n - 2$. A function $h : V(L_n) \to \{1,2,\dots,q+1\}$ is defined by $h(u_\alpha) = 3\alpha - 2,$ $\alpha = 1,2,\dots,n$,$h(v_\alpha) = 3\alpha - 1,$ $\alpha = 1,2,\dots,n$. The corresponding induced edge labels are $h^*(u_\alpha u_{\alpha+1}) = 3\alpha - 1, \alpha = 1,2,\dots,$ $n-1$,$h^*(v_\alpha v_{\alpha+1}) = 3\alpha,$ $\alpha = 1,2,\dots,n-1,$

$h^*(u_\alpha v_\alpha) = 3\alpha - 2, \alpha = 1,2,\dots,n$. Thus h^* is bijective. Therefore, L_n admits a square harmonic mean graph.

Illustration 2.12. The image below displays a square harmonic mean labeling of L_5.

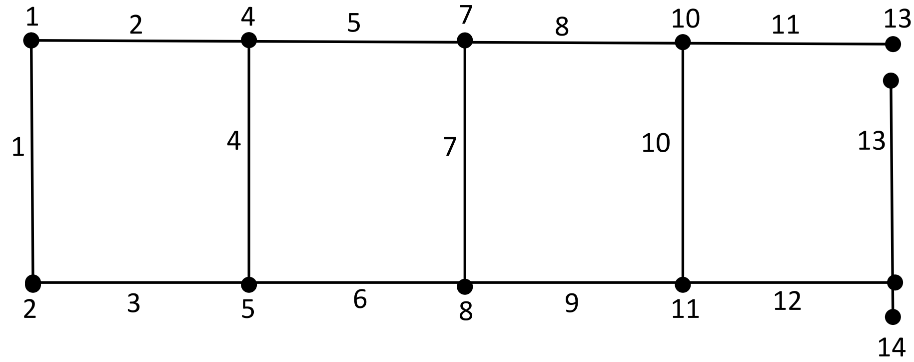

Figure VI. L_5

Theorem 2.13. Caterpillar graph admits a square harmonic mean graph.

Proof. Let G be a graph attained by joining a single edge to the two sides of each vertex of P_n.Let P_n be a path $u_1 u_2 \dots u_n$. Let v_α and w_α be the pendant vertices adjacent to u_α. Let $V(G) = \{u_\alpha, v_\alpha, w_\alpha : 1 \le \alpha \le n\}$ and $E(G) = \{u_\alpha u_{\alpha+1}, u_\alpha v_\alpha, u_n v_n, u_\alpha w_\alpha, u_n w_n : 1 \le \alpha \le n - 1\}$. Then$|V(G)| = 3n$ and $|E(G)| = 3n - 1$. A function $h : V(G) \to \{1,2,\dots,q+1\}$is defined by $h(u_\alpha) = 3\alpha - 1,\ \alpha = 1,2,\dots,n,\ \ h(v_\alpha) = 3\alpha - 2,\ \alpha = 1,2,\dots,n,\ \ h(w_\alpha) = 3\alpha, \alpha = 1,2,\dots,n$. The corresponding induced edge labels are $h^*(u_\alpha u_{\alpha+1}) = 3\alpha,\ \alpha = 1,2,\dots,n-1$, $h^*(u_\alpha v_\alpha) = 3\alpha - 2,\ \alpha = 1,2,\dots,n,\ \ h^*(u_\alpha w_\alpha) = 3\alpha - 1,\ \alpha = 1,2,\dots,n$. Thus h^* is bijective. Therefore, caterpillar graph admits a square harmonic mean graph.

Illustration 2.14. The image below displays a square harmonic mean labeling of caterpillar graph for n = 4.

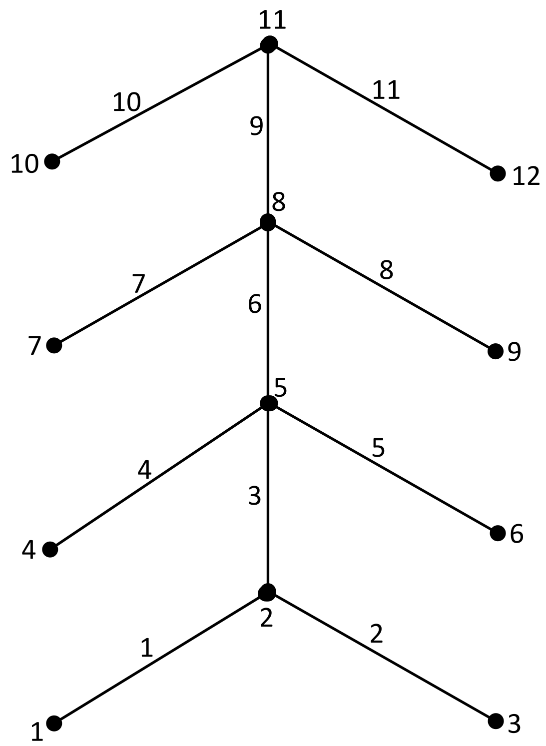

Figure VII.

Theorem 2.15. Olive trees $O_n$are square harmonic mean graph.

Proof. Let O_n be an olive tree with distinct paths P_α for $\alpha = 1,2,\dots,n$ joined at the end vertex v. The vertex v of degree n is root for O_n. Then O_n has $\frac{n(n+1)}{2} + 1$ vertices and $\frac{n(n+1)}{2}$ edges. A function $h : V(G) \to \{1,2,\dots,q+1\}$ is defined by $h(w) = 1$, $h(v_{11}) = w + 1$,$h(v_{\beta n}) = \beta + v_{(n-1)(n-1)}, 1 \le \beta \le n$, $h(v_{\beta(n-1)}) = \beta + v_{(n-2)(n-2)}, 1 \le \beta \le n-1$, $h(v_{\beta(n-2)}) = \beta + v_{(n-3)(n-3)}, 1 \le \beta \le n-2$, $h(v_{\beta(n-3)}) = \beta + v_{(n-4)(n-4)}, 1 \le \beta \le n-3$,….. The corresponding induced edge labels are $h^*(wv_{\beta 1}) = \beta, 1 \le \beta \le n$, $h^*(v_{\beta(n-2)}v_{\beta(n-1)}) = \beta + (n+4), 5 \le \beta \le n$, $h^*(v_{\beta(n-3)}v_{\beta(n-2)}) = \beta + (n+3), 4 \le \beta \le n$,$h^*(v_{\beta(n-4)}v_{\beta(n-3)}) = \beta + (n+1), 3 \le \beta \le n$, $h^*(v_{\beta(n-5)}v_{\beta(n-4)}) = \beta + n - 2$,$2 \le \beta \le n$,…. Thus h^* is bijective. Therefore, olive tree O_n admits a square harmonic mean graph.

Remark:

A simple Olive tree K_2 is square harmonic mean graph, since it is a complete graph with two vertices.

Illustration 2.16 The image below displays a square harmonic mean labeling of an olive tree O_5.

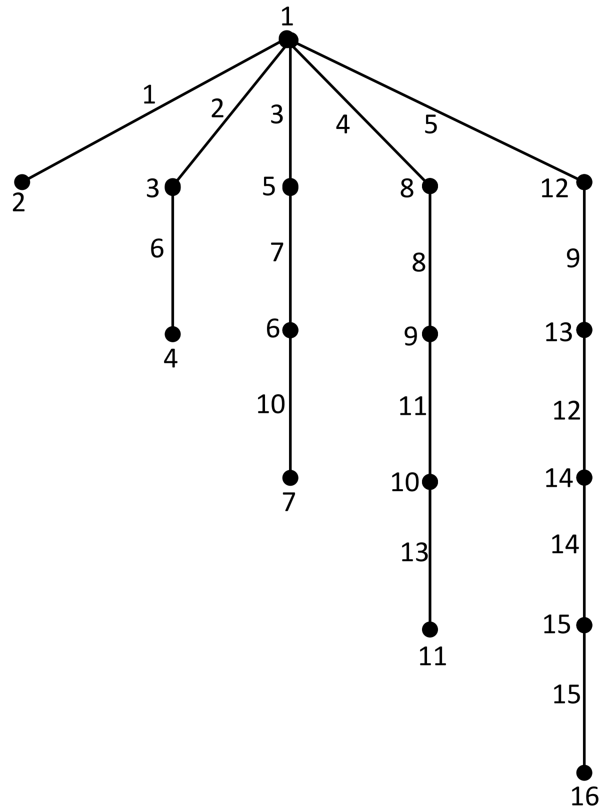

Figure VIII. O_5

Theorem 2.17. Triangular Snake $\mathrm{T_n}$admits a square harmonic mean graph.

Proof. Let $u_1u_2 \dots u_n$ be the path of length n and let $v_1v_2 \dots v_n$ be new vertices with joining the path u_α, $u_{\alpha+1}$respectively.Let$V(T_n) = \{u_\alpha, v_\alpha : \alpha = 1,2,\dots,n\}$and $E(T_n) = \{u_\alpha u_{\alpha+1}, u_\alpha v_\alpha, u_{\alpha+1}v_\alpha : \alpha = 1,2,\dots,n-1\}$. Then $|V(T_n)| = 2n - 1$and $|E(T_n)| = 3n - 3$. A function$h : V(T_n) \to \{1,2,\dots,q+1\}$is defined by$h(u_\alpha) = 3\alpha - 2$, $\alpha = 1,2,\dots,n$,$h(v_\alpha) = 3\alpha - 1, \alpha = 1,2,\dots,n-1$. The corresponding induced edge labels are $h^*(u_\alpha u_{\alpha+1}) = 3\alpha - 1, \alpha = 1,2,\dots,n-1$, $h^*(u_\alpha v_\alpha) = 3\alpha - 2, \alpha = 1,2,\dots,n-1$, $h^*(u_{\alpha+1}v_\alpha) = 3\alpha, \alpha = 1,2,\dots,n-1$.Thus h^* is bijective. Therefore,$T_n$admits a square harmonic mean graph.

Illustration 2.18. The image below displays a square harmonic mean labeling of$\mathrm{T_5}$.

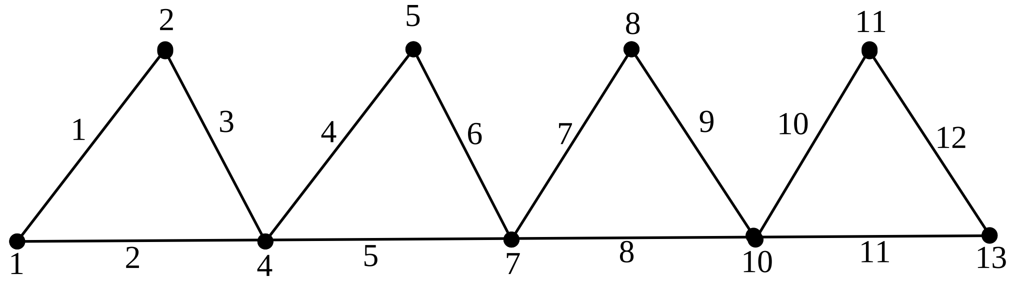

Figure IX.T_5

Theorem 2.19. Quadrilateral Snake $Q_n$admits a square harmonic mean graph.

Proof.$Q_n$is attained by attaching every pair of vertices $u_1\,u_2\,...\,u_n$ of a path P_n to another two new vertices v_α and $w_\alpha$respectively. Let$V(Q_n) = \{u_\alpha, u_n, v_\alpha, w_\alpha: \alpha = 1,2, \dots, n-1\}$and $E(Q_n) = \{u_\alpha u_{\alpha+1}, u_\alpha v_\alpha, u_{\alpha+1} w_\alpha, v_\alpha w_\alpha: \alpha = 1,2, \dots, n-1\}$.Then$|V(Q_n)| = 3n-2$ and $|E(Q_n)| = 4n-4$.A function$h : V(Q_n) \to \{1,2, \dots, q+1\}$is defined by$h(u_\alpha) = 4\alpha - 3, \alpha = 1,2, \dots, n$,$h(v_\alpha) = 4\alpha - 2, \alpha = 1,2, \dots, n-1$, $h(w_\alpha) = 4\alpha - 1, \alpha = 1,2, \dots, n-1$. The corresponding induced edge labels are$h^*(u_\alpha u_{\alpha+1}) = 4\alpha - 1, \alpha = 1,2, \dots, n-1$,$h^*(u_\alpha v_\alpha) = 4\alpha - 3,\ \alpha = 1,2, \dots, n-1$, $h^*(u_{\alpha+1} w_\alpha) = 4\alpha, \alpha = 1,2, \dots, n-1$,$h^*(v_\alpha w_\alpha) = 4\alpha - 2, \alpha = 1,2, \dots, n-1$.Thus h^* is bijective.Therefore, Quadrilateral Snake$Q_n$admits a square harmonic mean graph.

Illustration 2.20. The image below displays a square harmonic mean labeling of Quadrilateral SnakeQ_5.

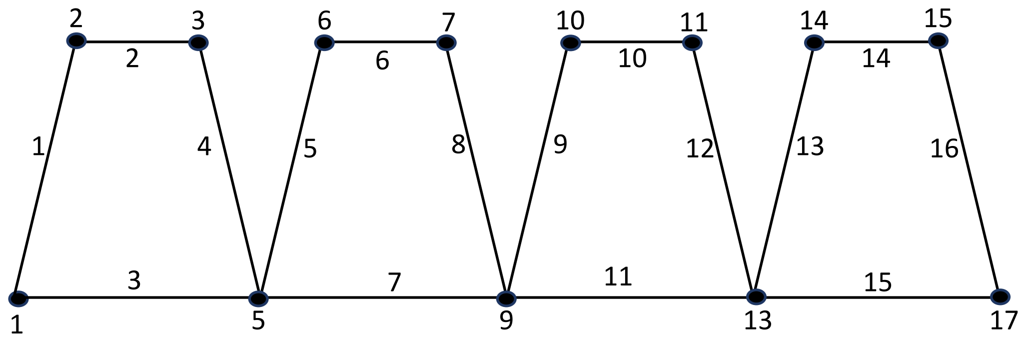

Figure X.Q_5

Theorem 2.21. Hexagonal snake HS_nadmits a square harmonic mean graph.

Proof. Let HS_n be a hexagonal snake graph. Consider a path $v_1 v_2 \dots v_n$of size n. Every edge of a path can be replaced by a cycle C_6.Then$|V(HS_n)| = 5n+1$ and $|E(HS_n)| = 6n$.A function$h : V(HS_n) \to \{1,2, \dots, q+1\}$is defined by$h(u_1) = 1$,$h(u_\alpha) = 6\alpha - 4$, $\alpha = 2, \dots, n$,$h(v_1) = 2$, $h(v_{\alpha+1}) = 6\alpha, \alpha = 1,2, \dots, n$, $h(w_\alpha) = 6\alpha - 3, \alpha = 1,2, \dots, n$,$h(x_\alpha) = 6\alpha - 2, \alpha = 1,2, \dots, n$, $h(y_\alpha) = 6\alpha - 1, \alpha = 1,2, \dots, n$. The corresponding induced edge labels are$h^*(u_\alpha v_\alpha) = 6\alpha - 5, \alpha = 1,2, \dots, n$, $h^*(u_\alpha u_{\alpha+1}) = 6\alpha - 2, \alpha = 1,2, \dots, n$, $h^*(v_\alpha w_\alpha) = 6\alpha - 4, \alpha = 1,2, \dots, n$, $h^*(x_\alpha w_\alpha) = 6\alpha - 3, \alpha = 1,2, \dots, n$, $h^*(x_\alpha y_\alpha) = 6\alpha - 1, \alpha = 1,2, \dots, n$, $h^*(y_\alpha y_{\alpha+1}) = 6\alpha, \alpha = 1,2, \dots, n$. Thus h^* is bijective. Therefore,hexagonal snake graphHS_nadmits a square harmonic mean graph.

Illustration 2.22. The image below displays a square harmonic mean labeling of HS_4.

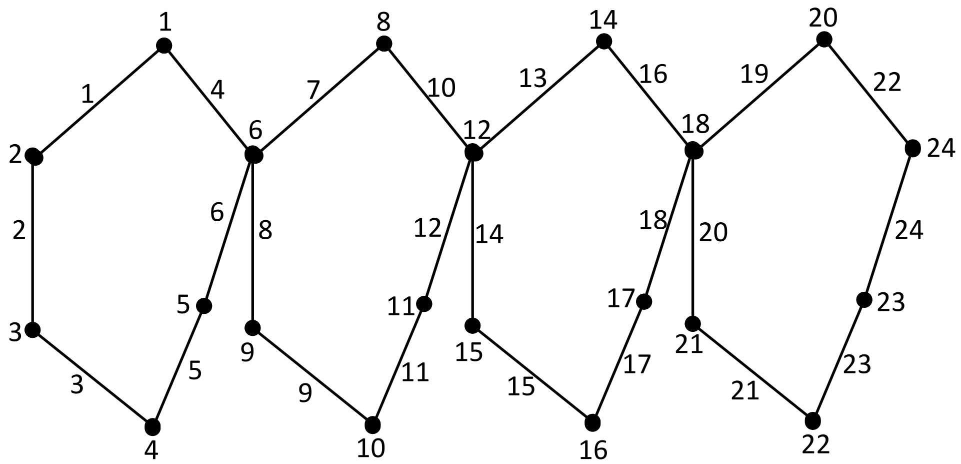

Figure XI.HS_4

3. Conclusion

In this paper we defined Square Harmonic mean labeling concept and showed how to label graphs like Path, Cycle, Comb, Star Graph, Crown Graph, Ladder Graph,Caterpillar, Olive Tree, Triangular Snake, Quadrilateral Snake and Hexagonal Snake Graph. We also provided illustrative examples for possible implementation of square harmonic mean labeling technique. It is possible to probe able results for several other graphs.

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