L(3,1)LABELING AND L'(3,1) LABELING OF MERGE GRAPH(C_3* $K_{1,n}$), SUBDIVISION OF THE EDGES OF THE STAR GRAPH, ($K_{1,n}$), AND L(3,1) LABELING OF TADPOLE GRAPH T(3,n) AND LILLY GRAPH I_n

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Abstract

$L(3,1)$ labeling is one of a particular model of frequency assignment problem of $L(h,k)$ labeling. A $L(3,1)$ labeling of a graph G is a function f from the vertex set $V(G)$ to the set of positive integers such that for any two vertices u,v if $d(u,v)$=1 then $|f(u)-f(v)| \geq 3$ and if $d(u,v)$=2 then $|f(u)-f(v)| \geq 1$. In $L(3,1)$ labeling, λ is the smallest positive integer which denotes the maximum label used.
$L'(3,1)$ labeling is an injective $L(3,1)$ labeling and λ' denotes the smallest positive integer which denotes the maximum label used in $L'(3,1)$ labeling.
In this paper, we consider $L(3,1)$ labeling and $L'(3,1)$ labeling of Merge graph$(C_3 * K_{1,n})$, Subdivision of the edges of the star graph $K_{1,n}$, and $L(3,1)$ labeling of Tadpole graph $T(3,n)$ and Lilly graph I_n.

Keywords: $L(3,1)$ labeling , $L'(3,1)$ labeling. 2010 Mathematics Subject Classification: 05C78

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1. Introduction

Graph Labeling is assigning labels to vertices or edges or both under certain conditions. The applications are found in communication networks, astronomy and so on. In wireless communication networks, we observe that the radio frequencies allotted to them are not adequate. The interference by unconstrained transmitters will disturb the communication. Hale.W [5] took up this problem in terms of graph labeling. Griggs and Robert proposed a variation in channel assignment problem. According to him any two transmitters which are close will receive different channels so as to avoid interference. $L(2,1)$ labeling is a result of this problem introduced by Griggs. J and Yeh. R [3]. L(3,1) labeling was introduced by Sumanto Ghosh and Anita Pal [8] whose definitions are as follows.

Definition 1 (L(2,1) labeling [3]).
Let $G = (V, E)$, $L(2,1)$ labeling (or distance two labeling) is a function $f: V(G) \to \{0,1, \dots k\}$ where k denotes the span of the graph, with the following conditions being satisfied

$$|f(x) - f(y)| \geq 2 \text{ if } d(x, y) = 1$$
$$|f(x) - f(y)| \geq 1 \text{ if } d(x, y) = 2$$

The largest number in $f(V)$ is the span of f. The λ number of G denoted as $\lambda(G)$ is the minimum span taken over all $L(2,1)$ labeling of G. Zero is taken as the minimum label in $L(2,1)$ labeling.

Definition 2 (L(3,1) labeling [8]).
Let the graph $G = (V, E)$, $L(3,1)$ labeling is a function f that assigns labels for every u, v belonging to the set of positive integers, if $d(u, v) = 1$ then $|f(u) - f(v)| \geq 3$ and if $d(u, v)$=2 then $|f(u) - f(v)| \geq 1$. $L(3,1)$ labeling number, $\lambda(G)$ is the smallest number λ with λ as the maximum label such that G has $L(3,1)$ labeling.
Definition 3 ($L'(3,1)$ labeling [8]).
$L'(3,1)$ labeling is an injective $L(3,1)$ labeling. $L'(3,1)$ labeling number $\lambda'(G)$ is the smallest number λ' with λ' as the maximum label such that G has $L'(3,1)$ labeling.
Definition 4 (Merge Graph [9]).
A *merge graph* $G_1 * G_2$ is formed from graphs G_1 and G_2 by merging a vertex of G_1 with a vertex of G_2.
Definition 5 (Subdivision of a graph [9]).
A graph obtained from G by a sequence of edge subdivisions is called a subdivision of a graph G.
Definition 6 (Tadpole Graph [4]).
$T(3, n)$ is a *Tadpole graph* in which any one vertex of cycle C_3 is attached to the path P_n.
Definition 7 (Lilly graph [2]).
$I_n, n \geq 2$ is the Lilly graph constructed using two star graphs $2K_{1,n}, n \geq 2$ and joining two paths $2P_n, n \geq 2$ which share a common vertex i.e

$$I_n = 2P_n + 2K_{1,n}$$

2. Main Results

$L(3,1)$ labeling number and $L'(3,1)$ labeling number of Merge graph$\left(C_3 * K_{1,n}\right)$, Subdivision of the edges of the star graph $K_{1,n}$ and $L(3,1)$ labeling of Tadpole graph $T(3.n)$ and Lilly graph I_n are determined in this section.

Theorem 2.1. *$L(3,1)$ labeling and $L'(3,1)$ labeling number of the Merge graph $(C_3 * K_{1,n})$ is $\lambda(C_3 * K_{1,n}) = \lambda'(C_3 * K_{1,n}) = n + 4$.*

Proof. Let the vertices of C_3 be u_0, u_1, u_2 and let the pendant vertices of $K_{1,n}$ be $u_3, u_4, \dots u_{n+2}$ such that $deg(u_0) = n + 2$. Also we observe that $diam(G) = 2$. Let u_0 be labelled as 0 such that u_1, u_2 receive the labels 3, 6 respectively and $u_3, u_4, \dots u_{n+2}$ are labeled as $4,5, \dots n + 4$ respectively. Suppose u_0 receives label other than 0 say u_0 as 1 then u_1 and u_2 should be 4 and 7 and $u_i's$, $i = 3,4, \dots$ cannot be 2 or 3 and hence it can take the labels $5,6,8 \dots n + 5$ which is not minimum. Also we observe that the labeling in each of the vertices are distinct.

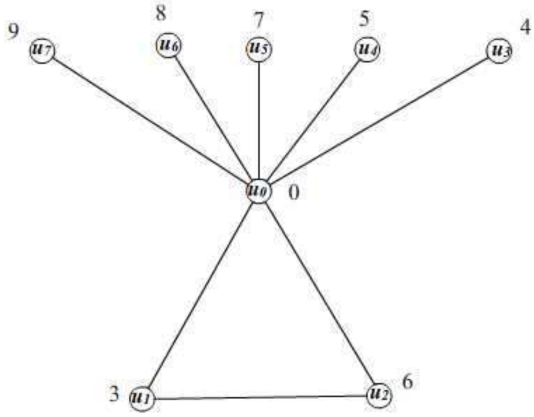

Figure 1 L(3,1) and L'(3,1) Labeling of Merge graph$(C_3 * K_{1,5})$

Hence it follows that

$$\lambda(C_3 * K_{1,n}) = \lambda'(C_3 * K_{1,n}) = n + 4$$

See Figure 1.

Theorem 2.2. *$L(3,1)$ labeling number of the subdivision of the edges of the star graph $K_{1,n}$ is*

$$\lambda\left(S(K_{1,n})\right) = n + 2 \; for \; all \; n > 3$$

.

Proof. Let u_0 be the root vertex of the subdivision of the edges of the star graph $K_{1,n}$. Let $u_1, u_2, \dots u_n$ be the vertices adjacent to u_0 and $u_1', u_2', \dots u_n'$ are the pendant vertices adjacent to $u_1, u_2, \dots u_n$ respectively. If u_i receives the label l then u_0 and u_i' should receive the label $\leq l - 3$ or $\geq l + 3$. Without loss of generality let $u_1 = 3$ then $u_0 = 0$ and $u_1' = 6$, since $d(u_1, u_2) = 2$. Let $u_2 = 4$ and hence $u_3, u_4, \dots u_n$ receive labels $5,6, \dots n + 2$. Also $u_i' = 1$, $i = 2,3, \dots n$ since it satisfies the condition $|f(u_i) - f(u_i')| \geq 1$. Therefore $\lambda(S(K_{1,n}) = n + 2$. See Figure 2.

Remark 1. $\lambda\left(S(K_{1,n})\right) = n + 3$ $(n = 2,3)$ *with u_0 taking the values either 1 or 4 or 5 when $n = 2$ and with u_0 taking the values either 0 or 1 or 5 or 6 when $n = 3$*

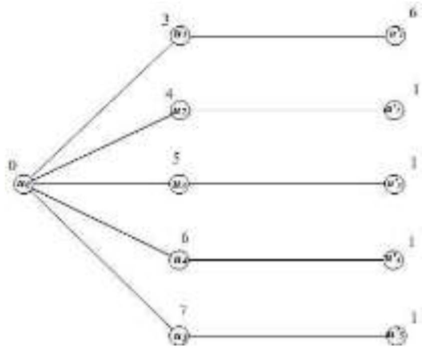

Figure 2: L(3,1) labeling of the subdivision of the edges of the star graph $K_{1,5}$

Theorem 2.3. $L'(3,1)$ *labeling number of the subdivision of the edges of the star graph is*

$$\lambda'\left(S(K_{1,n})\right) = 2n \text{ for all } n > 4 .$$

Proof. In view of Theorem 2.2, the vertices $u_3', u_4', \dots u_n'$ should receive different labels. Since u_3 receives the label 5, u_3' can be labelled with 2 and $u_4', u_5', \dots u_n'$ receive labels $n+4, n+5, \dots 2n$. Thus $\lambda'\left(S(K_{1,n})\right) = 2n$ $(u_0 = 0)$ for all $n > 4$. See Figure 3.

Remark 2. $\lambda'\left(S(K_{1,n})\right) = n + 3$ $(n = 2,3)$ *with* u_0 *taking the values either 0 or 1 and* $\lambda'\left(S(K_{1,4})\right) = 9$ *with* $u_0 = 0$

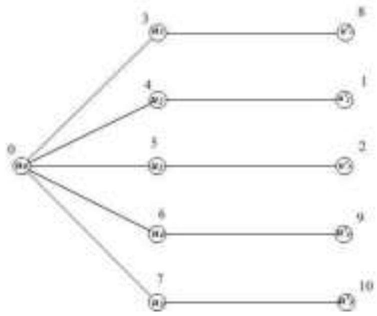

Figure 3 $L'(3,1)$ *labeling of the Subdivision of the edges of the star graph* $K_{1,5}$

Theorem 2.4. $L(3,1)$ *labeling number of the Tadpole graph* $T(3,n)$ *is*

$$\lambda\left(T(3,n)\right) = 6.$$

Proof. Let the vertices of C_3 be u, v, w such that the path P_n is attached with $u = u_0$ and the consecutive vertices in P_n are labelled as $u_1, u_2, \dots u_{n-1}$. The vertices of C_3 should receive the minimum labels as $l-3, l, l+3$. Let v and w be labelled as 0 and 3 and let u be labelled as 6, then u_1 and u_2 can receive the labels 1 and 4 respectively. Hence $u_3, u_4, u_5, u_6, u_7, u_8 \dots$ receive the labels 0,3,6,0,3,6, ... Therefore $f: V(G) \rightarrow \{0,1,2, \dots 6\}$ is defined as

$f(v) = 0$, $f(w) = 3$, $f(u) = 6$, $f(u_1) = 1$, $f(u_2) = 4$, and for $i \geq 3$

$$f(u_i) = \begin{cases} 0 & i \equiv 0 \bmod 3 \\ 3 & i \equiv 1 \bmod 3 \\ 6 & i \equiv 2 \bmod 3 \end{cases}$$

Thus $\lambda\left(T(3,n)\right) = 6$

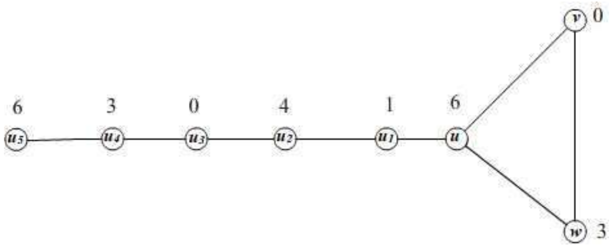

Figure 4 $L(3,1)$ *Labeling of Tadpole graph* $T(3,6)$

Theorem 2.5. $L(3,1)$ *labeling number of the Lilly graph* I_n *is*

$$\lambda(I_n) = 2n + 3, n \geq 4$$

Proof. Let the path vertices of I_n be labeled as $u_1, u_2, \ldots u_{2n-1}$ and the upper star with u_n as root vertex be labeled as $v_1, v_2, \ldots v_n$ and lower star with u_n as root vertex be labeled as $v_{n+1}, v_{n+2}, \ldots v_{2n}$. Without loss of generality, let u_n be labeled as 0, then $u_{n-1}, u_{n-2}, \ldots u_1$ are labeled as 6,3,0,6,3,0 ... and $u_{n+1}, u_{n+2}, \ldots$ are labeled as 3,6,0,3,6,0 .... Since $d(u_n, v_i) = 1$ and $d(v_i, u_j) \geq 2$, $(i, j = 1,2, \ldots n)$, v_1 v_2 can receive labels 4 and 5. Also $v_3, v_4, \ldots v_{2n}$ receive labels 7,8,9,10, ... $2n + 4$.
Thus $\lambda(I_n) = 2n + 4$ for $(n \geq 4)$

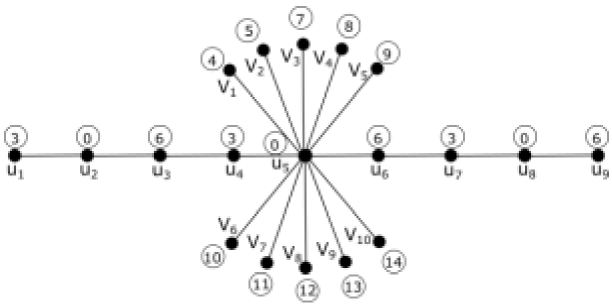

Figure 5 $L(3,1)$ *Labeling of Lilly graph* I_5

3. Conclusion

$L(3,1)$ labeling number and $L'(3,1)$ labeling number of Merge graph $C_3 * K_{1,n}$ and Subdivision of the edges of the star graph $K_{1,n}$ has been derived. $L(3,1)$ labeling number of Tadpole graph $T(3, n)$ and Lilly graph I_n is derived. For more graphs, work is under investigation.

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