THE FORCING METRIC DIMENSION OF JOIN OF GRAPHS

[1]S. Jency,[2]V. Mary Gleeta and [3]V.M. Arul Flower Mary
[1] Register Number:21113042092010, Research Scholar,
Department of Mathematics,
Holy Cross College (Autonomous), Nagercoil-629004, India.
email: jenijencyjs@gmail.com
[2]Assistant Professor, Department of Mathematics,
T.D.M.N.S. College, T.Kalikulam-627113, India.
email: gleetass@gmail.com
[3]Associate Professor, Department of Mathematics,
Holy Cross College (Autonomous), Nagercoil-629004, India.
email: arulflowermary@gmail.com
Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627 012,
Tamil Nadu, India

Abstract

Let $W = \{w_1, w_2, \dots \dots, w_k\}$ be an ordered subset of $V(G)$; then the metric representation of $v \in V(G)$ with respect to W is defined as the k- tuple $r(v/W) = d(v, w_1), d(v, w_2), \dots, d(v, w_k)$. The set W is called a resolving set of G if for all $u \neq v$ and $u, v \in V(G)$ satisfy $r(v/W) \neq r(u/W)$. A resolving set W of G with the minimum cardinality is the metric dimension of G and is denoted by $\dim(G)$. Any resolving with cardinality $\dim(G)$ is called dim-set of G or basis of G. Let W be a minimum resolving set of G. A subset $T \subseteq W$ is called a forcing subset for W if W is the unique minimum forcing resolving set containing T. A forcing subset for W of minimum cardinality is a minimum forcing subset of W. The forcing metric dimension of W denoted by $f_{dim}(W)$ is the cardinality of a minimum forcing subset of W. The forcing metric dimension of G, denoted by $f_{dim}(G)$, is $f_{dim}(G) = \min\{f_{dim}(W)\}$, where the minimum is taken over all minimum forcing resolving sets W in G. In this article, we determine the forcing metric dimension for join of two graphs.

Keywords: resolving set, metric dimension, forcing metric dimension, join of graphs.

AMS Subject Classification: 05C12.

1. Introduction and Preliminaries

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by n_1 and n_2 respectively. For basic graph theoretic terminology, we refer to [3]. Two vertices u and v are said to be adjacent if uv is an edge of G. Two edges of G are said to be adjacent if they have a common vertex. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. An u-v path of length $d(u, v)$ is called an u-v geodesic. These concepts were studied in [1,2, 9-13].

The join of two graphs G_1 and G_2denoted by $G_1 + G_2$ is a graph with vertex set $V(G_1) \cup V(G_2)$ and the edge set$E(G_1) \cup E(G_2) \cup \{uv/u \in V(G_1), v \in V(G_2)$. In the graph $G_1 + G_2$ each vertex of $G_1$is adjacent to the vertices of G_2 and $d(u_i, v_j) = 1; for u_i \in V(G_1), v_j \in V(G_2)$.

Let $W = \{w_1, w_2, \dots \dots, w_k\}$ be an ordered subset of $V(G)$; then the metric representation of$v \in V(G)$with respect to$W$ is defined as the k- tuple $r(v/W) = d(v, w_1), d(v, w_2), \dots, d(v, w_k)$. The set$W$ is called a resolving set of G if for all$u \neq v$and$u, v \in V(G)$satisfy$r(v/W) \neq r(u/W)$. A resolving set$W$ of$G$ with the minimum cardinality is the metric dimension of G and is denoted by $\dim(G)$. Any resolving with cardinality $\dim(G)$ is called dim-set of G or basis of G. A vertex v of a graph G is said to be *resolving vertex* of G if v belongs to every dim-set of G [4,5,19].

Let W be a minimum resolving set of G. A subset $T \subseteq W$ is called a forcing subset for Wif $W$is the unique minimum forcing resolving set containing T. A forcing subset for $W$of minimum cardinality is a minimum forcing subset of W. The forcing metric dimension of Wdenoted by $f_{dim}(W)$ is the cardinality of a minimum forcing subset of W. The forcing metric dimension of G, denoted by $f_{dim}(G)$, is $f_{dim}(G) = \min\{f_{dim}(W)\}$, where the minimum is taken over all minimum forcing resolving sets

Win G. The forcing metric dimension of a graph was introduced and studied in [6]. Then many authors studied the forcing concepts in [7, 8, 14-21]. In this article, we studied the forcing metric dimension in corona of two graphs.

Navigation can be studied in a graph structure framework in which the navigation agent moves from node to node of a graph space. The robot can locate itself by the presence of distinctly labeled landmark nodes in a graph space. If the robot knows its distances to a sufficiently large set of landmarks, its position on the graph is uniquely determined. This suggests the following problem: given a graph, what are the fewest number of landmarks needed, and where they should be located, so that the distance to the landmarks uniquely determines the robot's position on the graph? A minimum set of landmarks which uniquely determines the robot's position is called basis, and the minimum number of landmarks is called the metric dimension of graph. The following theorem is used in the sequel.

Theorem:1.1.[4] Let G be a connected graph and W be the set of all resolving vertices of G. Then $f_{dim}(G) \leq dim(G) - |W|$.

2. FORCING METRIC DIMENSION OF JOIN OF GRAPHS

In this section, we determine the forcing metric dimension for join of two graphs.

Theorem:2.1 Let P_{n_1} be a path of n_1 vertices and K_{n_2} be a complete graph with n_2 vertices. Then $f_{dim}(P_{n_1} + K_{n_2}) = \begin{cases} 4 & 2 \leq n_1 \leq 5 \\ n_2 + 1 & 6 \leq n_1 \leq 8. \\ n_2 & n_1 > 9 \end{cases}$

Proof: Let $V(P_{n_1}) = \{u_1, u_2, \dots \dots \dots, u_{n_1}\}$ and $V(K_{n_2}) = \{v_1, v_2, \dots \dots \dots, v_{n_2}\}$. We have the following cases

Case(i):$2 \leq n_1 \leq 5$. Let $W = \{v_1, v_2, \dots \dots, v_{n_2-1}, u_1, u_2\}$. Then

$r(v_{n_2}/W) = (1,1,1, \dots \dots, 1,1)$

$r(u_3/W) = (1,1,1, \dots \dots, 2,1)$

$r(u_4/W) = (1,1,1, \dots \dots, 2,2)$

Since representations are distinct, W is a resolving set for G and so $\dim(G) \leq n_2 + 1$. We prove that $\dim(G) = n_2 + 1$. On the contrary, suppose that $\dim(G) \leq n_2$. Then there exists a dim- set$W'$of Gsuch that $|W'| \leq n_2$. Without loss of generality, let $W' = \{v_1, v_2, \dots \dots, v_{n_2-1}, u_i\}$. Then $r\left(v_{n_2}/W\right) = (1,1,1, \dots \dots, 1) = r(u_i/W)$, Which is contrary. Therefore $\dim(G) = n_2 + 1$. We have to prove $f_{dim}(G) = 4$. Any dim-set W is of the form $W = S \cup \{u_i, u_j\}$, where $i \neq j$ and $S \subseteq V(K_{n_1})$ such that $|S| = n_2 - 1$.

Since any proper subset T with $|T| \leq 3$ is not a forcing subset of $W, f_{dim}(W) \geq 4$. Since W is unique the dim-set containing $\{v_1, v_2, u_1, u_1\}$, $f_{dim}(W) = 4$. Since this is true for all dim-sets W in G, $f_{dim}(G) = 4$.

Case(ii) For $6 \leq n_1 \leq 8$. Let $W = \{v_1, v_2, \dots \dots, v_{n_1-1}, u_2, u_4\}$. Then the metric representation of any vertex of $V(P_{n_1} + K_{n_2})/W$ with respect to W are

$r\left(v_{n_2}/W\right) = (1,2,1, \dots \dots, 1,1)$

$r(u_1/W) = (1,1,1, \dots \dots, 1,2)$

$r(u_3/W) = (1,1,1, \dots \dots, 1,1)$

$r(u_5/W) = (1,1,1, \dots \dots, 1,1,2)$

$r(u_6/W) = (1,1,1, \dots \dots, 1,2,2)$.

Since each representations are distinct, W is a resolving set of G. So that $\dim(G) \leq n_2 + 1$. We prove that $\dim(G) \leq n_2$. Then there exist a dim-set W' of G such that $|W'| \leq n_2$. Without loss of generality, let $W' = \{v_1, v_2, \dots \dots, v_{n_2-1}, u_2\}$. Then

$r(u_1/W) = (1,1,1, \dots \dots, 1,1)$

$r(u_3/W) = (1,1,1, \dots \dots, 1,1)$

Which is a contradiction. Therefore $\dim(G) = n_2 + 1$. Any dim-set S of G is of the term either.

(i)$S_1 \cup \{x, y\}$, where S_1contains $n_2 - 1$ elements and x, y are independent such that $S_1 \subseteq V(K_{n_2})$ and $x, y \in P_{n_1}$.

(ii) $S_2 \cup \{x, y, z\}$, where $S_2 \subseteq V(K_{n_2})$ such that $|S_2| = n_2 - 2$ and x, y, z are either independent or only two elements of $\{x, y, z\}$ are adjacent. Then $S_1 \cup \{x, y\}$ is a minimum forcing subset of S so that $f_{dim}(G) = n_1 + 1$.

Case(iii):$n_1 \geq 9$ and n_1 is even. Let $S \subseteq V(K_{n_2})$ such that $|S| = n_2 - 1$. Then $W_1 = S \cup \{u_2, u_4, \dots \dots, u_{n_1-2}\}$ and $W_2 = S \cup \{u_3, u_5, \dots \dots, u_{n_1-1}\}$ are the only $\dim_s -set$ of $G, f_{dims}(W_1) = f_{dims}(W_2) = n_2 - 1 + 1 = n_2$ so that $f_{dims}(G) = n_2$.

Case(iv):$n_1 \geq 9$ and n_1 is odd. Let $M \subseteq V(K_{n_2})$ such that $|M| + n_2 - 1$. Then $W_1 = M \cup \{u_2, u_4, \dots \dots, u_{n_1-1}\}$ and $W_2 = M \cup \{u_3, u_5, \dots \dots, u_{n_1}\}$ are the only $\dim_s -set$ of G, $f_{dims}(W_1) = f_{dims}(W_2) = n_2 - 1 + 1 = n_2$, so that $f_{dims}(G) = n_2$.

Theorem:2.2 Let C_{n_1} be a cycle of n_1 vertices and K_{n_2} be a complete graph with n_2 vertices. Then $f_{dim}(C_{n_1} + K_{n_2}) = 4$.

Proof: Let $V(P_{n_1}) = \{u_1, u_2, \dots \dots \dots, u_{n_1}\}$ and $V(K_{n_2}) = \{v_1, v_2, \dots \dots \dots, v_{n_2}\}$. We have the following cases

Case(i): $4 \leq n_1 \leq 6$. Let $M \subset V(K_{n_2})$ such that $|M| = n_2 - 1$. Then $W_i = M \cup \{u_i, u_{i+1}\}$ $(1 \leq i \leq n_1 - 1)$ and $W_{n_1} = M \cup \{u_{n_1}, u_1\}$ are the only $dim - sets$ of G so that $\dim(G) = n_2 + 1$. Since any two adjacent vertex of C_{n_1} belongs to more than one $dim - set$ of $G, f_{dim}(G) \geq 3$. Let T be a subset of G with $|T| = 3$, where $T = T_1 \cup T_2$ such that $T_1 \subset V(K_{n_2})$ and T_2 contains two adjacent vertices of G. Without loss of generality, let $T_1 = \{v_1\}, T_2 = \{u_1, u_2\}$ and $W = \{v_1, v_2, \dots \dots, v_{n_2-1}, u_1, u_2\}$. Therefore $T \subset W$. Then $W' = \{W - \{v_2\}\} \cup \{u_{n_2}\}$. Then $T \subset W'$. Which implies T is not a forcing subset of W. Therefore $f_{dim}(G) \geq 4$. Now $\{v_1, v_2, u_1, u_2\}$is a forcing subset of W and so $f_{dim}(W) = 4$. Since this true for all dim-sets W in $G, f_{dim}(G) = 4$.

Case(ii): If $n_1 \geq 7$. Let $W_{ij} = M \cup \{u_i, u_j\}, (1 \leq i \neq j \leq n_1)$ where $|i - j = 1|$ or $|i - j \geq 2|$. Then W_{ij} is a dim-sets of G so that $\dim(G) = n_2 + 2$. Since any two adjacent vertex of C_{n_1} belongs to more than one dim-set of $G, f_{dim}(G) \geq 3$. Let T be a subset of G with $|T| = 3$, where $T = T_1 \cup T_2$ such that $T_1 \subset V(K_{n_2})$ and T_2 contains two adjacent vertices of G. Without loss of generality, let $T_1 = \{v_1\}, T_2 = \{u_1, u_2\}$ and $W = \{v_1, v_2, \dots \dots, v_{n_2-1}, u_1, u_2\}$. Therefore $T \subset W$. Then $W' = \{W - \{v_2\}\} \cup \{u_{n_2}\}$. Then $T \subset W'$. Which implies T is not a forcing subset of W. Therefore $f_{dim}(G) \geq 4$. Now $\{v_1, v_2, u_1, u_2\}$is a forcing subset of W and so $f_{dim}(W) = 4$. Since this is true for all dim-sets W in $G, f_{dim}(G) = 4$.

Theorem:2.3 Let K_{1,n_1-1} be a star graph with n_1 vertices and K_{n_2} be a complete graph with n_2 vertices. Then

$$f_{dim}(K_{1,n_1-1} + K_{n_2}) = n_2 + 1.$$

Proof: Let $V(K_{1,n_1-1}) = \{x, u_1, u_2, \dots \dots, u_{n_1}\}$ and $V(K_{n_2}) = \{v_1, v_2, \dots \dots, v_{n_2}\}$. Let $W = \{v_1, v_2, \dots \dots, v_{n_2-1}, x, u_1, u_2\}$. Then

$r(v_{n_2}/W) = (1,1,1, \dots \dots ,1,1)$

$r(u_3/W) = (1,1,1, \dots \dots ,1,2)$

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$r(u_{n_1-1}/W) = (1,1,1, \dots \dots ,2,2)$.

Since each representations distinct, W is a resolving set of G, and so $\dim(G) \leq n_2 + 2$. We prove that $\dim(G) \leq n_2 + 2$. On the contrary, suppose that $\dim(G) \leq n_2 + 1$.Then there exist a dim-set of G such that $|S'| \leq n_2 + 1$. Then there exists a vertex $y \in V(G)$ such that $y \notin S'$. First assume that $y \in V(K_{1,n_1-1})$. Hence there exists $z \in V(K_{1,n_1-1})$ such that $z \notin S'$ and $z \neq y$. Then $r(y/S') = r(z/S')$, which is a contradiction. If $y \in V(K_{n_2})$, by the similar way, we get a contradiction. Therefore $\dim(G) = n_2 + 2$. We prove that $f_{\dim}(G) = n_2 + 1$. By Theorem 1.1, $f_{dim}(G) \leq \dim(G) - |x| = n_2 + 1$. Since any dim - set S of G the form$S = \{x\} \cup X \cup Y$, where $X \subset V(K_{1,n_1-1}) - \{x\}$ and $y \in V(K_{n_2})$ suh that $|x| = 2$ and $|y| = n_2 - 1$. Suppose that $f_{dim}(G) \leq n_2$. Then there exist a for every subset $T \subseteq S$ such that $|T| \leq n_2$. Let u be a vertex of K_{1,n_1-1} suvh that $u \in T$ and $u \neq x$. Since $n_1 - 1 \geq 2$, there exists $v \in K_{1,n_1-1}$ such that $v \neq u$and $v \neq x$and $v \notin T$. Let $S_2 = \{S_1 - \{u\}\} \cup \{v\}$. Then $T \subset S_2$. Hence it follows that S_1 is not a unique dim-set of G contrary T, which is a contradiction. Therefore $f_{dim}(G) = n_2 + 1$.

Theorem:2.4 Let F_{1,n_1} be a fan graph with n_1 vertices and K_{n_2} be a complete graph with n_2 vertices. Then

$$f_{dim}(F_{1,n_1} + K_{n_2}) = n_2 + 1.$$

Proof: Let $V(F_{1,n_1}) = \{x, u_1, u_2, \dots \dots, u_{n_1}\}$ and $V(K_{n_2}) = \{v_1, v_2, \dots \dots, v_{n_2}\}$. Let $W = \{v_1, v_2, \dots \dots, v_{n_2-1}, x, u_1, u_2\}$. Then

$r(v_{n_2}/W) = (1,1,1, \dots \dots ,1,1)$

$r(u_3/W) = (1,1,1, \dots \dots ,2,1)$

$r(u_4/W) = (1,1,1, \dots \dots ,2,2)$

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$r(u_{n_1}/W) = (1,1,1, \dots \dots ,1,2)$.

Since each representations are distinct, W is a resolving set for G, and so $\dim(G) \leq n_2 + 2$. We prove that $\dim(G) \leq n_2 + 2$. On the contrary, suppose that $\dim(G) \leq n_2 + 1$. Then there exist a dim-set of G such that $|S'| \leq n_2 + 1$. Then there exists a vertex $y \in V(G)$ such that $y \notin S'$. First assume that $y \in V(F_{1,n_1})$. Hence there exists $z \in V(F_{1,n_1})$ such that $z \notin S'$ and $z \neq y$. Then $r(y/S') = r(z/S')$, which is a contradiction. If $y \in V(K_{n_2})$. By the similar way, we get a contradiction. Therefore $\dim(G) = n_2 + 2$. We prove that $f_{\dim}(G) = n_2 + 1$. By Theorem 1.1, $f_{dim}(G) \leq \dim(G) - |x| = n_2 + 1$. Since any dim-set S of G the form $S = \{x\} \cup X \cup Y$, where $X \subset V(F_{1,n_1}) - \{x\}$ and $y \in V(K_{n_2})$ suh that $X = \{u_i, u_j\}$, $|i - j| = 1$ and $|y| = n_2 - 1$. Suppose that $f_{dim}(G) \leq n_2$. Then there exists a for every subset $T \subseteq S$ such that $|T| \leq n_2$. Let u be a vertex of F_{1,n_1} such that $u \in T$ and $u \neq x$. Since $n_1 - 1 \geq 2$, there exists $v \in F_{1,n_1}$ such that $v \neq u$ and $v \neq x$ and $v \notin T$. Let $S_2 = \{S_1 - \{u\}\} \cup \{v\}$. Then $T \subset S_2$. Hence it follows that S_1 is not a unique dim-set of G contrary T, which is a contradiction. Therefore $f_{dim}(G) = n_2 + 1$.

Theorem:2.5 Let W_{n_1} be a wheel graph with n_1 vertices and K_{n_2} be a complete graph with n_2 vertices. Then

$$f_{dim}(W_{n_1} + K_{n_2}) = n_2 + 1.$$

Proof: Let $V(W_{n_1}) = \{x, u_1, u_2, \dots \dots , u_{n_1}\}$ and $V(K_{n_2}) = \{v_1, v_2, \dots \dots , v_{n_2}\}$. Let $W = \{v_1, v_2, \dots \dots , v_{n_2 - 1}, x, u_1, u_2\}$. Then

$r(v_{n_2}/W) = (1,1,1, \dots \dots ,1,1)$

$r(u_3/W) = (1,1,1, \dots \dots ,2,1)$

$r(u_4/W) = (1,1,1, \dots \dots ,2,2)$

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$r(u_{n_1}/W) = (1,1,1, \dots \dots, 1,2)$.

Since each representations are distinct, W is a resolving set for G, and so $\dim(G) \leq n_2 + 2$. We prove that $\dim(G) \leq n_2 + 2$. On the contrary, suppose that $\dim(G) \leq n_2 + 1$. Then there exist a dim-setof G such that $|S'| \leq n_2 + 1$. Then there exists a vertex $y \in V(G)$ such that $y \notin S'$. First assume that $y \in V(W_{n_1})$. Hence there exists $z \in V(W_{n_1})$ such that $z \notin S'$ and $z \neq y$. Then $r(y/S') = r(z/S')$, which is a contradiction. If $y \in V(K_{n_2})$. By the similar way, we get a contradiction. Therefore $\dim(G) = n_2 + 2$. We prove that $f_{\dim}(G) = n_2 + 1$. By Theorem 1.1, $f_{dim}(G) \leq \dim(G) - |x| = n_2 + 1$. Since any dim-set S of G the form $S = \{x\} \cup X \cup Y$, where $X \subset V(W_{n_1}) - \{x\}$ and $y \in V(K_{n_2})$ suh that $X = \{u_i, u_j\}$, $|i - j| = 1$ and $|y| = n_2 - 1$. Suppose that $f_{dim}(G) \leq n_2$. Then there exist a for every subset $T \subseteq S$ such that $|T| \leq n_2$. Let u be a vertex of W_{n_1} such that $u \in T$ and $u \neq x$. Then since $n_1 - 1 \geq 2$, there exists $v \in W_{n_1}$ such that $v \neq u$ and $v \neq x$ and $v \notin T$. Let $S_2 = \{S_1 - \{u\}\} \cup \{v\}$. Then $T \subset S_2$. Hence it follows that S_1 is not a unique dim-set of G contrary T, which is a contradiction. Therefore $f_{dim}(G) = n_2 + 1$.

Theorem:2.6 Let G be a connected graph of order $n_1 \geq 3$. Then $f_{dim}(G \odot K_1) = n_1 - 1$.

Proof: Let $V(G) = \{v_1, v_2, \dots \dots, v_{n_1}\}$ and $V(G \odot K_1) = \{u_1, u_2, \dots \dots, u_{n_1}\}$. Let $W = \{u_1, u_2 \dots \dots, u_{n_1 - 1}\}$. Then

$r(v_1/W) = (1,2, \dots \dots, 1,2)$

$r(v_2/W) = (2,1, \dots \dots, 1,2)$

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$r(v_{n_1}/W) = (2,2, \dots \dots, 2,1,2)$

$r(u_{n_1}/W) = (1,1,1, \ldots \ldots, 1,1)$.

Since each representations are distinct, W is a resolving set for G, and so $\dim(G) \leq n_1 - 1$. We prove that $\dim(G) = n_1 - 1$. On the contrary, suppose that $\dim(G) \leq n_1 - 2$. Then there exists a dim- set S such that $|S| \leq n_1 - 2$. Let $x, y \in V$ such that $x, y \notin S$. If $S = \{u_1, u_2, \ldots \ldots, u_{n_1-2}\}$, then $r(u_{n_1-3}/S) = r(u_{n_1-2}/S) = (1,1, \ldots \ldots, 1,1)$. Therefore $S \not\subset W$. If $S \subset \{v_1, v_2, \ldots \ldots, v_{n_1}\}$, let $S = \{v_1, v_2, \ldots \ldots, v_{n_1-2}\}$. Then $r(v_{n_1-3}/S) = r(v_{n_1-2}/S) = (2,2, \ldots \ldots, 2,1)$. Therefore S contains at least one element from $\{v_1, v_2, \ldots \ldots, v_{n_1}\}$ and at least one element from $\{u_1, u_2, \ldots \ldots, u_{n_1}\}$. Without loss of generality, let $S = \{v_1, v_2, \ldots \ldots, v_{n_1-4}, u_1, u_2\}$. Then $r(v_{n_1-1}/S) = r(v_{n_1}/S) = (3,3, \ldots \ldots, 3,2,2)$. Which is a contradiction. By the similar way, if $S = \{v_1, v_2, \ldots \ldots, v_{n_1-2}\}$, then $r(v_{n_1-3}/S) = r(v_{n_1-2}/S) = (1,1, \ldots \ldots, 1,1)$. Which is a contradiction. Therefore the dim- sets are

(i) For $1 \leq i \leq n_1$, $W_i = \{v_1, v_2, \ldots \ldots, v_{n_1}\} - \{x\}$ where $x \in \{v_1, v_2, \ldots \ldots, v_{n_1}\}$.

(ii) For $1 \leq j \leq n_1$, $W_j = \{u_1, u_2, \ldots \ldots, u_{n_1}\} - \{y\}$ where $y \in \{u_1, u_2, \ldots \ldots, u_{n_1}\}$.

For $1 \leq i \leq n_1$, W_i is the unique dim-set of G contains $\{v_1, v_2, \ldots \ldots, v_{n_1-1}\}$ is a forcing subset of$W$ so that $f_{dim}(W_i) = n_1 - 1, (1 \leq i \leq n_1)$. For $1 \leq j \leq n_1, W_j$ is the unique dim- set of G contains $\{u_1, u_2, \ldots \ldots, u_{n_1-1}\}$ is a forcing subset of$W$ so that $f_{dim}(W_j) = n_1 - 1, (1 \leq j \leq n_1)$. Therefore $f_{dim}(G) = n_1 - 1$.

3. Conclusions

In this article we studied the forcing metric dimension for join of a graph. We extend this concept to other distance related parameters in graphs.

4. Acknowledgement

We are thankful to the referees for their constructive and detailed comments and suggestions which improved the paper overall.

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